Level Set Methods

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http://www.cab.u-szeged.hu/~kato/variational/

Introduction

- Developed by
  - Stanley Osher
    - http://www.math.ucla.edu/~sjo/
  - J. A. Sethian
    - http://math.berkeley.edu/~sethian/
- J. A. Sethian: Level Set Methods and Fast Marching Methods. *Cambridge University Press, 1999*.

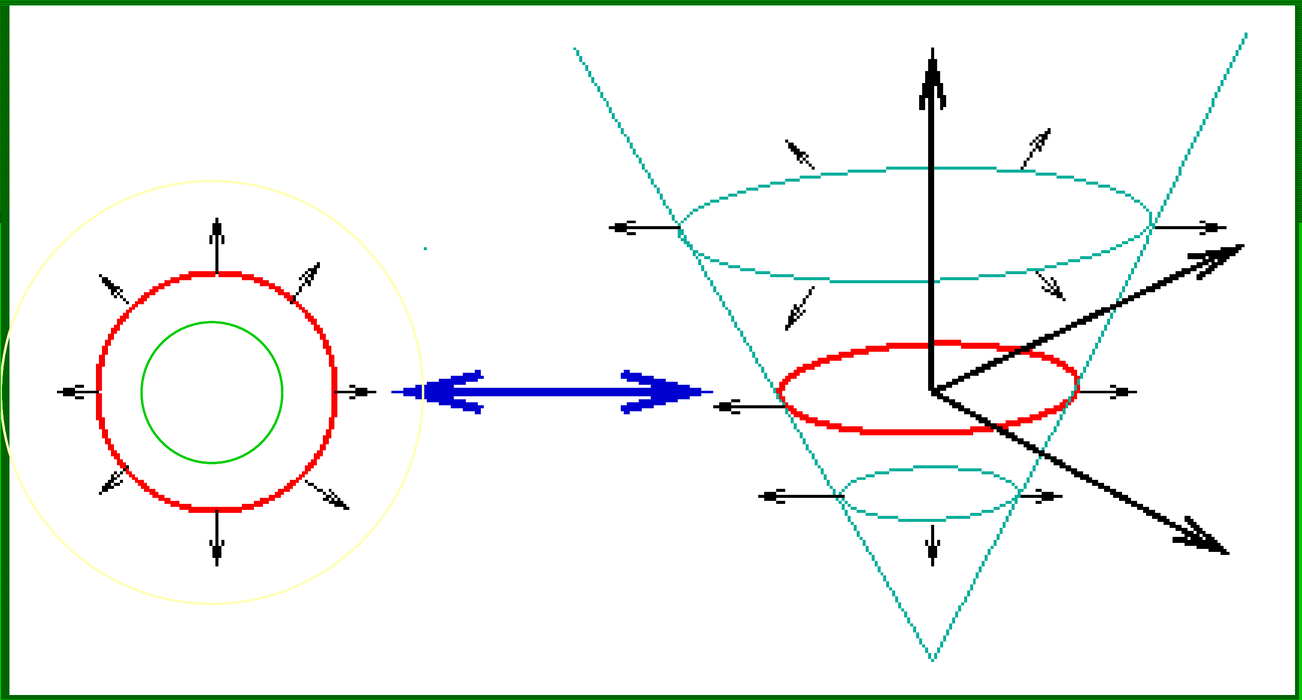
What is it?

- It is a generic numerical method for evolving fronts in an implicit form.
  - It handles topological changes of the evolving interface
  - Define problem in 1 higher dimension
    - Seems crazy but it well worth the extra effort…
- Use an implicit representation of the contour ***C*** as the zero level set of higher dimensional function ϕ ***- the level set function***

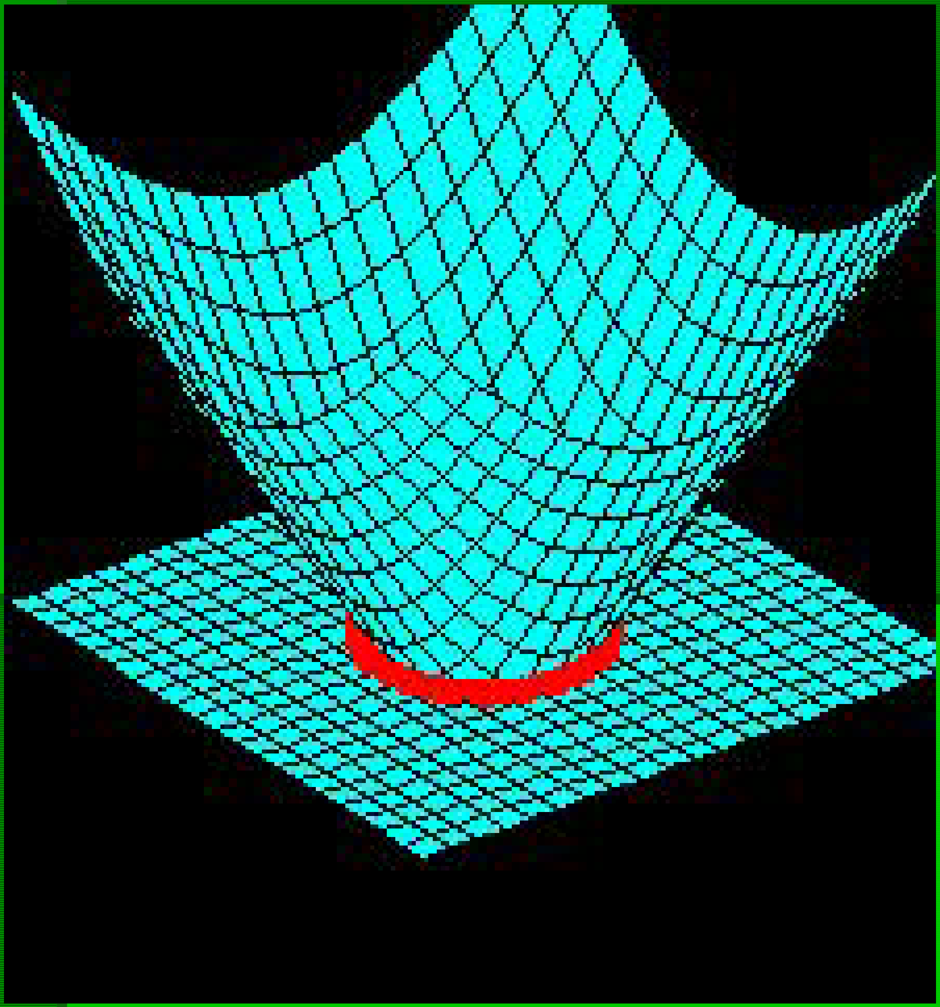
$$\phi(C) = 0$$

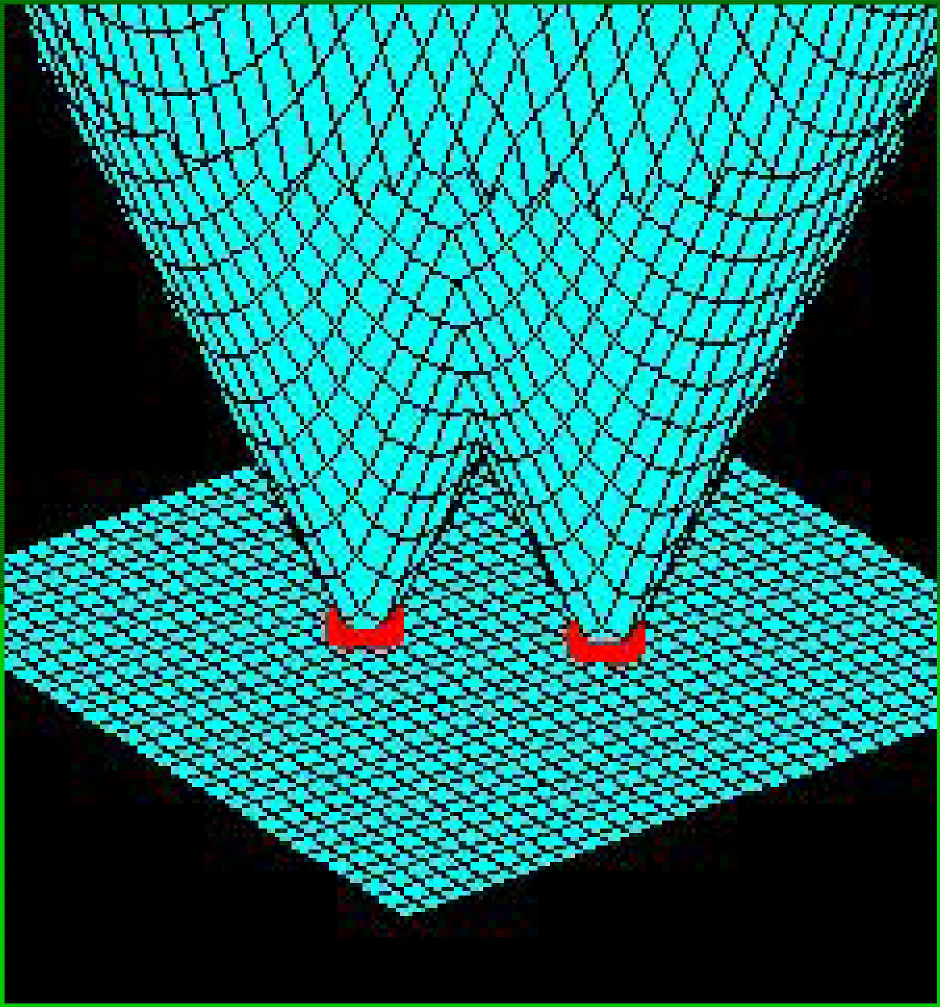
How it works?

- Move the level set function, $\phi(\mathbf{x},\mathbf{y},\mathbf{t})$, so that it rises, falls, expands, etc.
- Contour = cross section at **z** = 0



Implicit curve evolution





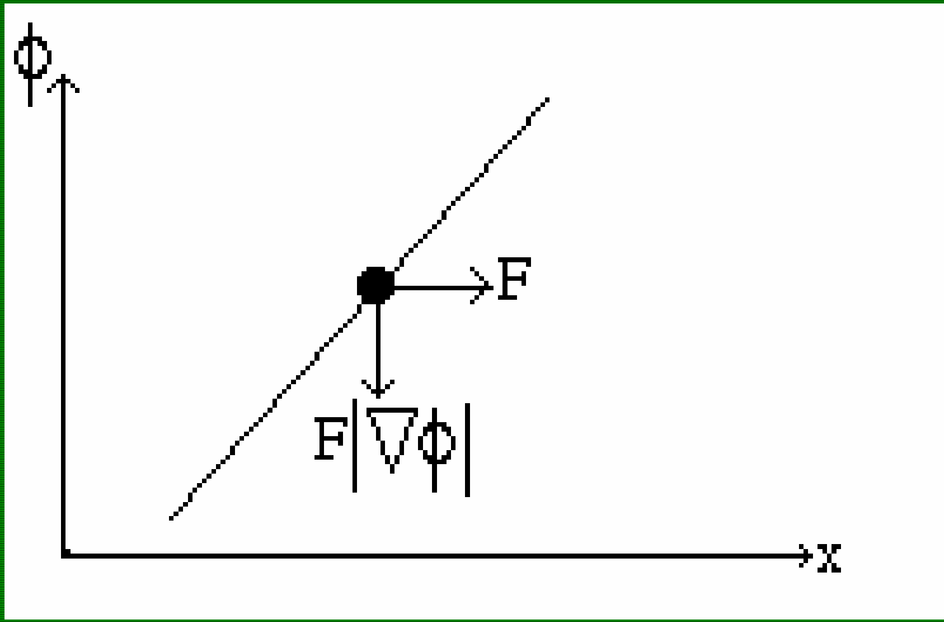
Images taken from the level set website: http://math.berkeley.edu/~sethian/Explanations/level_set_explain.html/

How to Move the Level Set Surface?

- Define a velocity field ***F***, that specifies how contour points move in time
  - Based on application-specific physics such as time, position, normal, curvature, image gradient magnitude
- Build an initial value for the level set function, $\phi(x,y,t=0)$, based on the initial contour position
- Adjust ϕ over time; current contour defined by $\phi(x(t), y(t), t) = 0$

The Level Set Evolution Equation

- Manipulate ϕ to indirectly move ***C***:



$$\phi(C) = 0$$

$$\frac{d\phi(C)}{dt} = \frac{\partial C}{\partial t} \cdot \nabla \phi + \frac{\partial \phi}{\partial t} = 0$$

$$\frac{\partial \phi}{\partial t} = -F \left| \nabla \phi \right|$$

where ***F*** is the speed function normal to the curve

Example: an expanding circle

- Level Set representation of a circle:

$$\phi(x,y)=\sqrt{x^2+y^2}-r$$

  - Setting ***F = 1*** causes the circle to expand uniformly
  - Observe that ***∇ϕ = 1*** almost everywhere (by choice of representation), so we obtain the level set evolution equation:

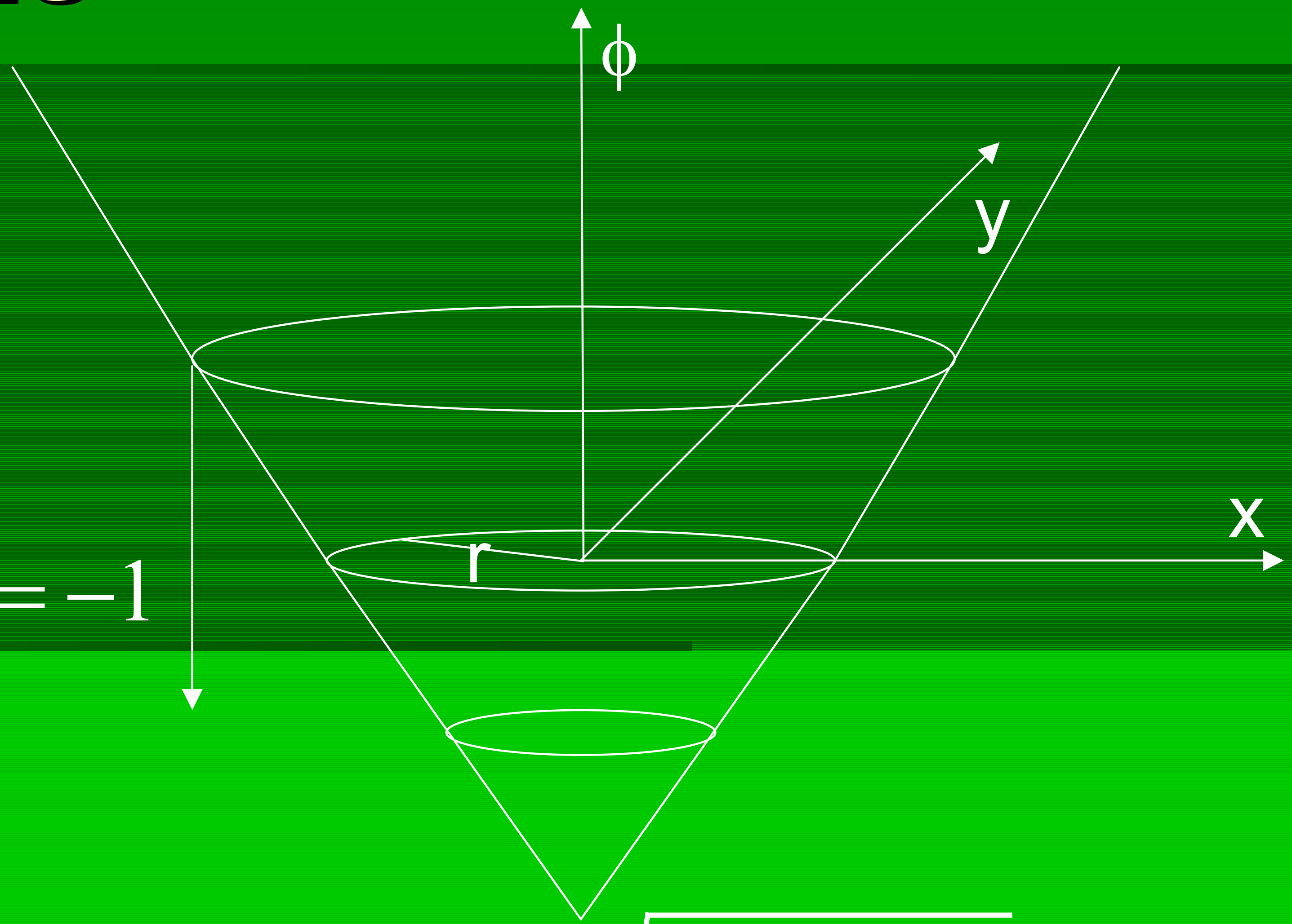
$$\frac{\partial\phi}{\partial t}=-1$$

- Explicit solution:

$$\phi(x,y,t)=\sqrt{x^2+y^2}-r-t$$

which means that the circle has radius ***r + t*** at time ***t***, as expected!

Example: an expanding circle

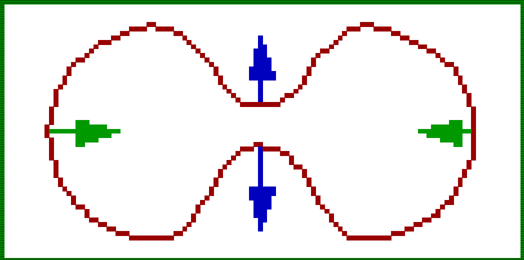


$$\frac{\partial \phi}{\partial t} = -1$$

$$\phi(x, y) = \sqrt{x^2 + y^2} - r$$

Motion under curvature

- What about more complicated shapes?



- ***Motion under curvature*: each piece of the curve moves perpendicular to the curve with speed proportional to the curvature.**
  - **Since the curvature can be either positive or negative (depending on whether the curve is turning clockwise or counterclockwise), some parts of the curve move outwards while others move inwards.**

Motion under curvature

- A famous theorem in differential geometry (proved in the 90's), says that:
  - any simple closed curve moving under its curvature collapses to a circle and then disappears.

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Level Set Segmentation

- Since the choice of ϕ is somewhat arbitrary, we can choose a signed distance function from the contour.
  - This distance function is negative inside the curve and positive outside.
  - A distance function is chosen because it has unit gradient almost everywhere and so is smooth.
- By choosing a suitable speed function ***F***, we may segment an object in an image

Level Set Segmentation

- The standard level set segmentation speed function is:

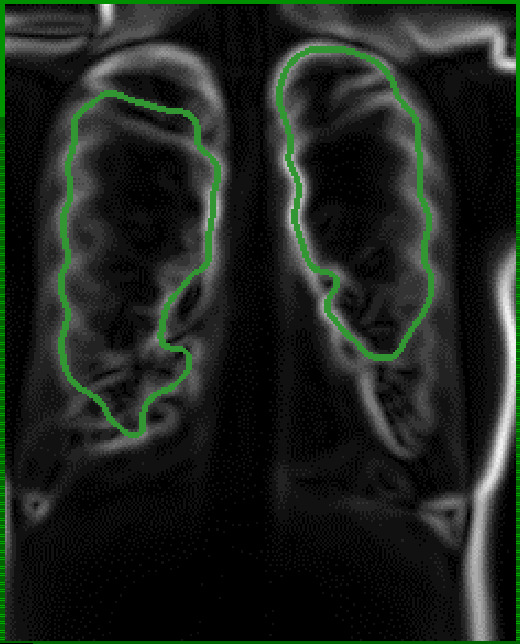
$$F = 1 - \varepsilon\kappa + \beta\left(\nabla\phi \cdot \nabla\left|\nabla I\right|\right)$$

- The ***1*** causes the contour to inflate inside the object
- The ***-$\varepsilon\kappa$*** (viscosity) term reduces the curvature of the contour
- The final term (edge attraction) pulls the contour to the edges
- Imagine this speed function as a balloon inflating inside the object. The balloon is held back by its edges, and where there are holes in the boundary it bulges but is halted by the viscosity ε.

Segmentation Example

Lung x-ray

Viscosity 5



Viscosity 2

Viscosity 0.5

Some more examples

Level Set Method

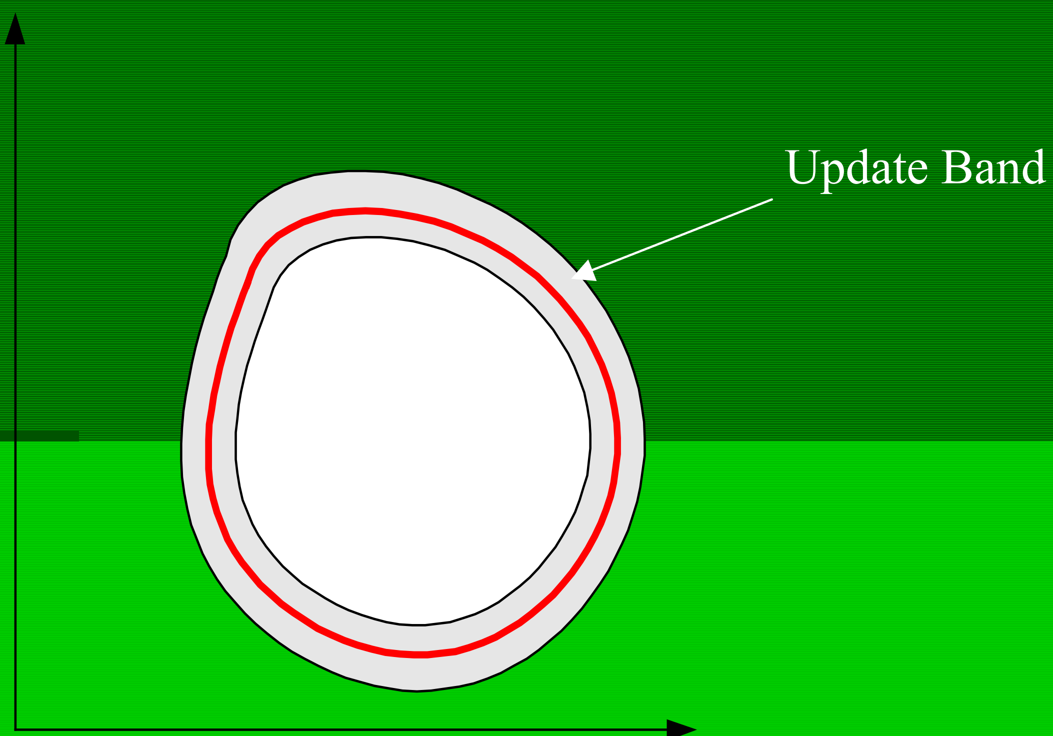
- Update equation:

$$\phi_i^{n+1} = \phi_i^n - \Delta t\left(\sqrt{D_i^{-x}} + \sqrt{D_i^{+x}}\right)$$

  - where D_i^{-x} and D_i^{+x} are forward and backward differences.
- Algorithm is computationally expensive
- Can be done in a more efficient manner:
  - Narrow Band Level Set Method
  - Fast Marching Methods

Narrow Band Level Set Method

- The efficiency comes from updating the speed function.
- We do not need to update the function over the whole image or volume.
- Update over a narrow band (2D) or tube (3D)



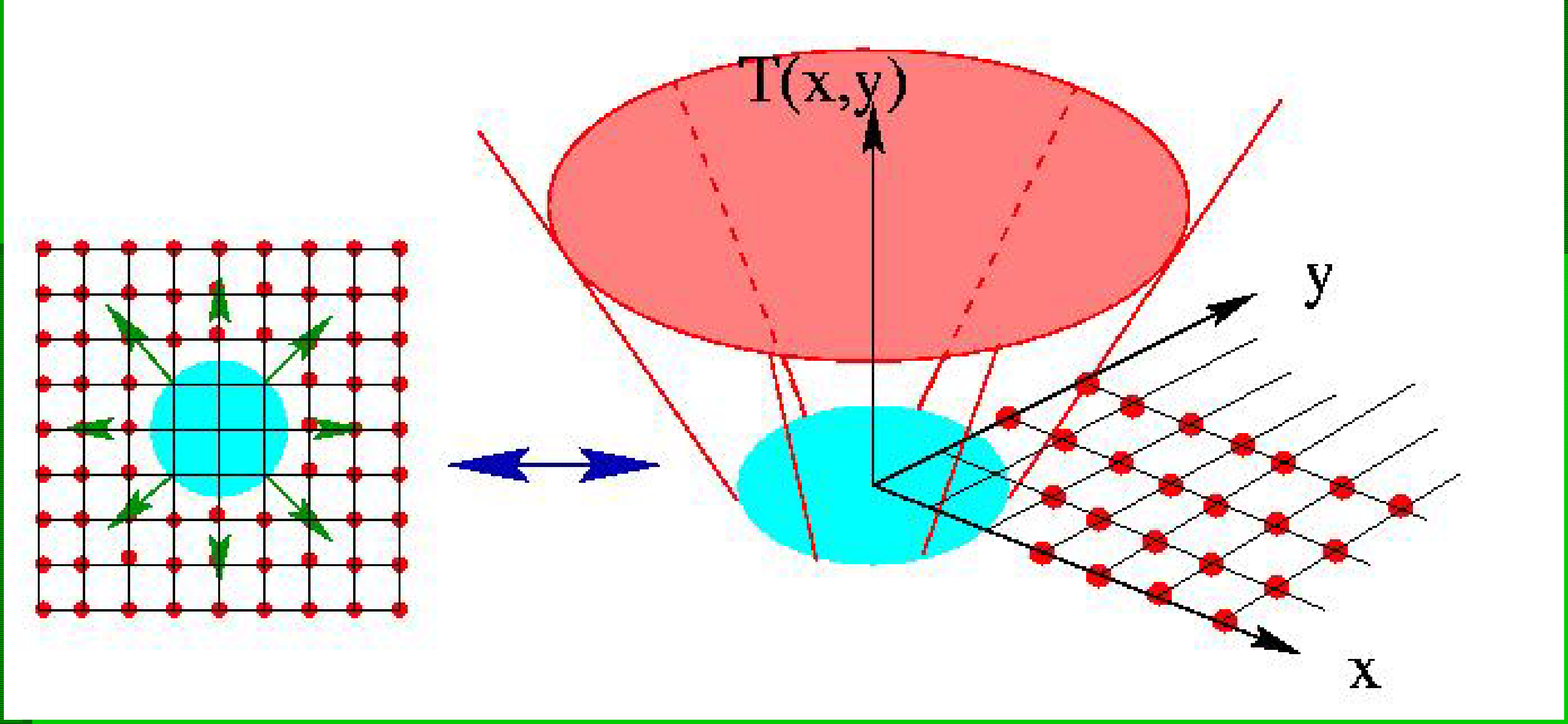
Fast Marching Method

- Proposed by J. Sethian in 1996
- Special case that assumes the velocity field *F* never changes sign.
  - That is, contour is either always expanding or always shrinking
- Can convert problem to a stationary formulation on a discrete grid where the contour is guaranteed to cross each grid point at most once

$$|\nabla T(x,y)|F = 1, \quad T = 0 \ \text{on} \ \Gamma$$

Fast Marching Method

- Compute *T(x,y)* = time at which the contour crosses grid point ***(x,y)***
- At any height ***T***, the surface gives the set of points reached at time ***T***



Fast Marching Algorithm

- Construct the arrival time surface *T(x,y)* incrementally:
  1. Build the initial contour
  2. Incrementally add on to the existing surface the part that corresponds to the contour moving with speed ***F***
- Builds level set surface by scaffolding the surface patches farther and farther away from the initial contour

Fast Marching