Applications of Linear Programming

lecturer: András London

University of Szeged
Institute of Informatics
Department of Computational Optimization

Lecture 2

Solving LP: slack variables

Let us consider the following LP:

$$\begin{aligned} \text{Max} \quad z = 3x_1 + 2x_2 \\ x_1 + x_2 &\leq 80 \\ 2x_1 + x_2 &\leq 100 \\ x_1 &\leq 40 \\ x_1, x_2 &\geq 0 \end{aligned}$$

To change an inequality to an equation, we add a new **non-negative variable** called a **slack variable**.

$$\begin{aligned} \text{Max} \quad z = 3x_1 + 2x_2 \\ x_1 + x_2 + x_3 &= 80 \\ 2x_1 + x_2 + x_4 &= 100 \\ x_1 + x_5 &= 40 \\ x_1, \ldots, x_5 &\geq 0 \end{aligned}$$

Solving LP:

Now we have

$$\text{Max} \quad z = 3x_1 + 2x_2$$

$$\begin{aligned} x_1 + x_2 + x_3 &= 80 \\ 2x_1 + x_2 + x_4 &= 100 \\ x_1 + x_5 &= 40 \\ x_1, \dots, x_5 &\geq 0 \end{aligned}$$

Express the slack variables from the individual equations

$$\begin{array}{rcrcrcr} x_3 & = & 80 & - & x_1 & - & x_2 \\ x_4 & = & 100 & - & 2x_1 & - & x_2 \\ x_5 & = & 40 & - & x_1 & & \\ \hline z & = & 0 & + & 3x_1 & + & 2x_2 \end{array}$$

This is called a **dictionary**.

Solving LP: dictionary

In the general case we have...

$$
\begin{array}{ccccccccccc}
a_{11}x_1 & + & a_{12}x_2 & + & \dots & + & a_{1n}x_n & + & x_{n+1} & = & b_1 \\
a_{21}x_1 & + & a_{22}x_2 & + & \dots & + & a_{2n}x_n & + & x_{n+2} & = & b_2 \\
 & & & & \vdots & & & & & & \\
a_{m1}x_1 & + & a_{m2}x_2 & + & \dots & + & a_{mn}x_n & + & x_{n+m} & = & b_m \\
\hline
c_1x_1 & + & c_2x_2 & + & \dots & + & c_nx_n & & & = & z
\end{array}
$$

...and the dictionary is

$$
\begin{array}{ccccccccccc}
x_{n+1} & = & b_1 & - & a_{11}x_1 & - & a_{12}x_2 & - & \dots & - & a_{1n}x_n \\
x_{n+2} & = & b_2 & - & a_{21}x_1 & - & a_{22}x_2 & - & \dots & - & a_{2n}x_n \\
 & & & & & \vdots & & & & & \\
x_{n+m} & = & b_m & - & a_{m1}x_1 & - & a_{m2}x_2 & - & \dots & - & a_{mn}x_n \\
\hline
z & = & & & c_1x_1 & + & c_2x_2 & + & \dots & + & c_nx_n
\end{array}
$$

Terminology

Decision variables: variables of the original LP (given in standard form) $(x_1, x_2, \ldots, x_n)$

Slack variables: new non-negative variables to construct the dictionary $(x_{n+1}, x_{n+2}, \ldots, x_{n+m})$

Basic variables variables on the left-hand side of the constraint equalities of the dictionary

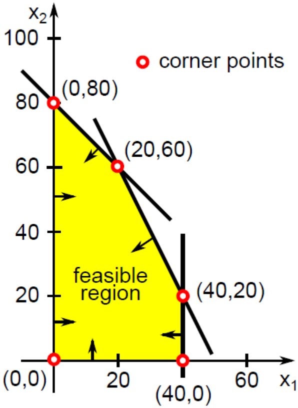
Non-basic variables variables on the right-hand side of the constraint equalities of the dictionary

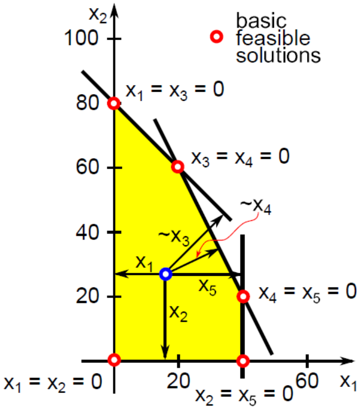
Basic solution vector x, such that the non-basic-variables are zero (and the basic variables are constant values of the respective equations on the right-hand side)

Feasible basic solution a basic solution which is feasible, i.e. $b_i \geq 0$ $i = 1, 2, \ldots, m$ is satisfied.

Dictionary: basic solution

$$\begin{array}{lrcrcl} \max & 3x_1 & + & 2x_2 & & \\ & x_1 & + & x_2 & \leq & 80 \\ & 2x_1 & + & x_2 & \leq & 100 \\ & x_1 & & & \leq & 40 \\ & & & x_1, x_2 & \geq & 0 \end{array} \quad \longrightarrow \quad \begin{array}{lrcrcrcrcrcl} \max & 3x_1 & + & 2x_2 & & & & & & & & \\ & x_1 & + & x_2 & + & x_3 & & & & & = & 80 \\ & 2x_1 & + & x_2 & & & + & x_4 & & & = & 100 \\ & x_1 & & & & & & & + & x_5 & = & 40 \\ & & & & & & & & x_1, x_2, x_3, x_4, x_5 & & \geq & 0 \end{array}$$





Dictionary: basic solution

Important remark: Basic solutions are precisely the corner points of the feasible region.

Recall that we have discussed that to find an optimal solution to an LP, it suffices to find a best solution among all corner points. The above tells us how to compute them – they are the basic feasible solutions.

Now we suppose that $b_1 \geq 0, \ldots, b_m \geq 0$, therefore the basic solution of the initial dictionary is

$$x = (0, 0, \ldots, 0, b_1, b_2, \ldots, b_m)$$

is a feasible solution.

Simplex algorithm

- Algorithm: **iterative search of the optimal solution**
- **Tries to find a new dictionary** such that:
  1. Every two consecutive dictionaries are equivalent
  2. In each iteration the value of the objective function is larger or equal, than in case the previous dictionary
  3. The basic solution is feasible in each iteration
- How to find a new dictionary?
- How to find it, such that the assumptions above would be satisfied?
- How do we know that a basic solution is optimal?
- Is there an optimal solution for every LP?

Simplex algorithm

Pivot step: calculate a new feasible by changing the roles of a basic and non-basic variable (i.e. re-arranging a constraint equation)

Incoming variable: a non-basic variable that become a basic variable of the new dictionary in a simplex iteration

Outgoing variable: a basic variable that become a non-basic variable of the new dictionary in a simplex iteration

Two dictionaries are equivalent: if the two have the same set of solutions with equal objective function values for the same solutions

Proposition 1. *Each dictionary is equivalent to the original system.*

Simplex algorithm

How to find a new dictionary? Example:

$$\begin{array}{rcrcrcrcr} x_4 & = & 4 & - & x_1 & - & 2x_2 & & \\ x_5 & = & 6 & - & x_1 & & & - & 4x_3 \\ x_6 & = & 2 & & & - & 2x_2 & + & 2x_3 \\ \hline z & = & 10 & + & 3x_1 & + & 4x_2 & - & x_3 \end{array}$$

- The value of the objective function increases if x_1 or x_2 increases in the basic solution
- The value of the objective function decreases if x_3 increases in the basic solution
- We can increase either x_1 or x_2...

Simplex algorithm

How to find a dictionary, such that the assumptions above would be satisfied?

$$\begin{array}{rcrcrcrcr} x_4 & = & 4 & - & x_1 & - & 2x_2 & & \\ x_5 & = & 6 & - & x_1 & & & - & 4x_3 \\ x_6 & = & 2 & & & - & 2x_2 & + & 2x_3 \\ \hline z & = & 10 & + & 3x_1 & + & 4x_2 & - & x_3 \end{array}$$

- How much can we increase x_2 before a (dependent) variable becomes negative?

$$x_4 \geq 0 \quad \Rightarrow \quad 4 - 2x_2 \geq 0 \quad \Leftrightarrow \quad 2 \geq x_2$$

$$x_6 \geq 0 \quad \Rightarrow \quad 2 - 2x_2 \geq 0 \quad \Leftrightarrow \quad 1 \geq x_2$$

- This can be streamlined into the simple "ratio" test.

Simplex algorithm

How do we know that a basic solution is optimal?

$$\begin{array}{rcrcrcrcr} x_4 & = & 4 & - & x_1 & - & 2x_2 & & \\ x_5 & = & 6 & - & x_1 & & & - & 4x_3 \\ x_6 & = & 2 & & & - & 2x_2 & + & 2x_3 \\ \hline z & = & & - & 3x_1 & - & 4x_2 & - & x_3 \end{array}$$

No more improvement possible $\rightarrow$ **optimal solution**

Theorem 1. *If there is no positive c_j $(j = 1, 2, \ldots, n + m)$ constant in the objective function and there is no negative b_i $(i = 1, 2, \ldots, m)$ constant in the constraint equations, than the basic solution of the dictionary is optimal.*

Proof. *Homework*

■

Simplex algorithm

Is there an optimal solution for every LP?

$$
\begin{array}{rcrcrcrcr}
x_4 & = & 4 & + & x_1 & - & 2x_2 & & \\
x_5 & = & 6 & + & x_1 & & & - & 4x_3 \\
x_6 & = & 2 & & & - & 2x_2 & + & 2x_3 \\
\hline
z & = & & & 3x_1 & - & 4x_2 & - & x_3
\end{array}
$$

We can make x_1 arbitrarily large and thus make z arbitrarily large $\rightarrow$ unbounded LP

Unbounded LP: if the objective function of the maximization (minimization) LP problem can be arbitrarily large (small)

Simplex algorithm

Preparation: find a starting feasible solution/dictionary

1. Convert to the canonical form (constraints are equalities) by adding slack variables $x_{n+1}, \ldots, x_{n+m}$
2. Construct a starting dictionary - express slack variables and objective function z
3. If the resulting dictionary is feasible, then we are done with preparation
   If not, try to find a feasible dictionary using the **Phase I. method** (next lecture).

Simplex algorithm

Simplex step (maximization LP): try to improve the solution

1. **(Optimality test)**: If **no variable** appears with a **positive** coefficient in the equation for z

   $\longrightarrow$ STOP, current solution is **optimal**

   - set non-basic variables to zero
   - read off the values of the basic variables and the objective function z
     $\longrightarrow$ Hint: the values are the constant terms in respective equations
   - report this (optimal) solution

2. Else pick a variable x_i having positive coefficient in the equation for z

   $x_i \equiv$ *incoming* variable

3. Ratio test: in the dictionary, find an equation for a variable x_j in which

   - x_i appears with a negative coefficient $-a$
   - the ratio $\frac{b}{a}$ is smallest possible

   (where b is the constant term in the equation for x_j)

4. If no such such x_j exists $\longrightarrow$ stop, no optimal solution, report that **LP is unbounded**

5. Else $x_j \equiv$ outgoing variable $\longrightarrow$ construct a new dictionary by *pivoting*:

   - express x_i from the equation for x_j,
   - add this as a new equation,
   - remove the equation for x_j,
   - substitute x_i to all other equations (including the one for z)

6. Repeat from 1.

Degeneracy

$$\begin{array}{rcl} x_4 &=& 1 \qquad\qquad\qquad\qquad - 2x_3 \\ x_5 &=& 3 - 2x_1 + 4x_2 - 6x_3 \\ x_6 &=& 2 + x_1 - 3x_2 - 4x_3 \\ \hline z &=& \quad\;\; 2x_1 - x_2 + 8x_3 \end{array}$$

Pivoting: x_3 enters, ratio test: $x_4 : \dfrac{1}{2} = 1/2$, $x_5 : \dfrac{3}{6} = 1/2$, $x_6 : \dfrac{2}{4} = 1/2 \longrightarrow$ any of x_4, x_5, x_6 can be chosen

$\to$ we choose x_4 to leave, $x_3 = \frac{1}{2} - \frac{1}{2}x_4$

$$\begin{array}{rcl} x_3 &=& \frac{1}{2} \qquad\qquad\qquad\quad - \frac{1}{2}x_4 \\ x_5 &=& \quad - 2x_1 + 4x_2 + 3x_4 \\ x_6 &=& \qquad\; x_1 - 3x_2 + 2x_4 \\ \hline z &=& 4 + 2x_1 - x_2 - 4x_4 \end{array}$$

setting $x_1 = x_2 = x_4 = 0$ yields $x_3 = \frac{1}{2}$, $x_5 = 0$, $x_6 = 0$

now x_1 enters, and x_5 leaves (the only choice), $x_1 = 2x_2 - \frac{3}{2}x_4 - \frac{1}{2}x_5$

$$\begin{array}{rcl} x_1 &=& \qquad\; 2x_2 + \frac{3}{2}x_4 - \frac{1}{2}x_5 \\ x_3 &=& \frac{1}{2} \qquad\quad\; - \frac{1}{2}x_4 \\ x_6 &=& \quad - x_2 + \frac{7}{2}x_4 - \frac{1}{2}x_5 \\ \hline z &=& 4 + 3x_2 - x_4 - x_5 \end{array}$$

setting $x_2 = x_4 = x_5 = 0$ yields $x_1 = 0$, $x_3 = \frac{1}{2}$, $x_6 = 0$

$\longrightarrow$ **same solution** as before

if some basic variable is zero, then the basic solution is **degenerate**

Degeneracy

Problem: several dictionaries may correspond to the same (degenerate) solution.

The simplex rule may cycle, it is possible to go back to the same dictionary if we are not careful enough when choosing the incoming/outgoing variables.

Bland's rule: From possible options, choose an incoming (outgoing) variable x_k with smallest subscript k.

Theorem. Simplex method using Bland's rule is guaranteed to terminate in a finite number of steps.

Alternative: **lexicographic rule** – choose as outgoing variable one whose row is lexicographically smallest (when divided by the constant term) – the coefficients in the objective function are guaranteed to strictly increase lexicographically

Two-phase simplex method

The dictionary for **LP in standard (maximization) form**

$$x_{n+i} = b_i - \sum_{j=1}^{n} a_{ij} x_j \qquad i = 1, 2, \ldots, m$$

$$\overline{}$$

$$z = \sum_{j=1}^{n} c_j x_j$$

- If every $\boldsymbol{b_i \geq 0}$ $i = 1, 2, \ldots, m$, then we can run the simplex algorithm
- **But if not?**

Phase I.

$$\begin{array}{lrcrcrl} \text{Max} \quad z = & x_1 & - & x_2 & + & x_3 & \\ & 2x_1 & - & x_2 & + & x_3 & \leq 4 \\ & 2x_1 & - & 3x_2 & + & x_3 & \leq -5 \\ & -x_1 & + & x_2 & - & 2x_3 & \leq -1 \\ & & & x_1, & & x_2, x_3 & \geq 0 \end{array}$$

The corresponding dictionary:

$$\begin{array}{rcrcrcrcr} x_4 & = & 4 & - & 2x_1 & + & x_2 & - & 2x_3 \\ x_5 & = & -5 & - & 2x_1 & + & 3x_2 & - & x_3 \\ x_6 & = & -1 & + & x_1 & - & x_2 & + & 2x_3 \\ \hline z & = & & & x_1 & - & x_2 & + & x_3 \end{array}$$

This is **not feasible** since $x_5, x_6 < 0$ in the basic solution.

Phase I

Idea Introduce one new artificial variable x_0 and a new objective $w = -x_0$:

$$
\begin{array}{llrcrcrcrl}
\text{Max} & w = & & & & & & & -x_0 & \\
& & 2x_1 & - & x_2 & + & x_3 & - & x_0 & \leq 4 \\
& & 2x_1 & - & 3x_2 & + & x_3 & - & x_0 & \leq -5 \\
& & -x_1 & + & x_2 & - & 2x_3 & - & x_0 & \leq -1 \\
& & & & & & x_1, x_2, & & x_3, x_0 & \geq 0
\end{array}
$$

Introduce x_4, x_5, x_6 slack variables as before:

$$
\begin{array}{llrcrcrcrcrl}
\text{Max} & w = & & & & & & & -x_0 & & & \\
& & 2x_1 & - & x_2 & + & x_3 & - & x_0 & + & x_4 & = 4 \\
& & 2x_1 & - & 3x_2 & + & x_3 & - & x_0 & + & x_5 & = -5 \\
& & -x_1 & + & x_2 & - & 2x_3 & - & x_0 & + & x_6 & = -1 \\
& & & & & & & & & & x_i & \geq 0
\end{array}
$$

Phase I.

- consider the inequality whose right-hand side is most negative (in this case 2nd inequality)
- this inequality has an associated slack variable (x_5), remove this variable from our set $\rightarrow \{x_4, x_6\}$
- add x_0 in place of the removed variable $\rightarrow \{x_0, x_4, x_6\}$

$$\begin{array}{rcrcrcrcrcr} x_0 & = & 5 & + & 2x_1 & - & 3x_2 & + & x_3 & + & x_5 \\ x_4 & = & 9 & & & - & 2x_2 & & & + & x_5 \\ x_6 & = & 4 & + & 3x_1 & - & 4x_2 & + & 3x_3 & + & x_5 \\ \hline w & = & -5 & - & 2x_1 & + & 3x_2 & - & x_3 & - & x_5 \end{array}$$

and this is **feasible**!

Phase I.

Theorem *The standard LP has a feasible solution if and only if $w = 0$ is the optimal solution of the auxiliary problem.*

Proof.

- Suppose that x is a feasible solution of the original problem.
- Then $(x_0 = 0, x)$ is an optimal solution of the auxiliary problem and $w\left((x_0 = 0, x)\right) = 0$
- Conversely, suppose that 0 is the optimum of the auxiliary problem taken in x^*
- Then $x_0^* = 0$, and leaving x_0^* from x^* we obtain a feasible solution of the original problem

∎

Two-pahse simplex method

Steps of Phase I.

1. If the basic solution of the corresponding dictionary of the LP in is feasible, then go to Phase II. (simplex algorithm)
2. If not, associate the auxiliary problem and prepare its initial feasible dictionary
3. Solve the auxiliary problem with the simplex algorithm
4. If the optimum < 0, then there is no feasible solution of the original problem
5. If the optimum $= 0$, then we get a dictionary that is equivalent with the dictionary of the original LP and its basic solution is feasible

Steps of Phase II

1. Run the simplex algorithm staring from the feasible dictionary obtained by Phase I.

Two-pahse simplex method: example

The initial dictionary:

$$\begin{array}{rcrcrcr} x_3 & = & -5 & - & x_1 & + & x_2 \\ x_4 & = & 6 & - & x_1 & - & x_2 \\ \hline z & = & & & 2x_1 & + & x_2 \end{array}$$

The auxiliary problem:

$$\begin{array}{rcrcrcrcr} x_3 & = & -5 & - & x_1 & + & x_2 & + & x_0 \\ x_4 & = & 6 & - & x_1 & - & x_2 & + & x_0 \\ \hline w & = & & & & & & - & x_0 \end{array}$$

Two-pahse simplex method: example

right-hand side is most negative: $x_3 = \cdots \Rightarrow$: use x_0 here as basic variable

$$\begin{array}{rcrcrcrcr}
x_0 & = & 5 & + & x_1 & - & x_2 & + & x_3 \\
x_4 & = & 11 & & & - & 2x_2 & + & x_3 \\
\hline
w & = & -5 & - & x_1 & + & x_2 & - & x_3
\end{array}$$

Incoming: x_2, outgoing: x_0:

$$\begin{array}{rcrcrcrcr}
x_2 & = & 5 & + & x_1 & - & x_0 & + & x_3 \\
x_4 & = & 1 & - & 2x_1 & + & 2x_0 & - & x_3 \\
\hline
w & = & & & & - & x_0 & &
\end{array}$$

Optimum: $x_0 = 0$, $w = 0 \Rightarrow$ leave x_0-t, back to the original objective function

Two-pahse simplex method: example

The obtained dictionary is equivalent to the initial one:

$$
\begin{array}{rcrcrcr}
x_2 & = & 5 & + & x_1 & + & x_3 \\
x_4 & = & 1 & - & 2x_1 & - & x_3 \\
\hline
z & = & & & 2x_1 & + & x_2
\end{array}
$$

We get (x_2 is changed to $5 + x_1 + x_3$ in the right-hand side)

$$
\begin{array}{rcrcrcr}
x_2 & = & 5 & + & x_1 & + & x_3 \\
x_4 & = & 1 & - & 2x_1 & - & x_3 \\
\hline
z & = & 5 & + & 3x_1 & + & x_3
\end{array}
$$

Running the simplex algorithm, the solution can be read from

$$
\begin{array}{rcrcrcr}
x_2 & = & 5.5 & - & 0.5x_4 & + & 0.5x_3 \\
x_1 & = & 0.5 & - & 0.5x_4 & - & 0.5x_3 \\
\hline
z & = & 6.5 & - & 1.5x_4 & - & 0.5x_3
\end{array}
$$

Fundamental theorem of linear programming

Theorem. *For an arbitrary linear program in standard form, the following statements are true:*

- *If there is no optimal solution, then the problem is either infeasible or unbounded.*
- *If a feasible solution exists, then a basic feasible solution exists.*
- *If an optimal solution exists, then a basic optimal solution exists.*

Proof.

- The Phase I algorithm either proves that the problem is infeasible or produces a basic feasible solution
- The Phase II algorithm either discovers that the problem is unbounded or finds a basic optimal solution
- These statements depend, of course, on applying a variant of the simplex method that does not cycle, which we now know to exist.

∎

Summary

The process consists of two steps

1. Find a **feasible** solution (or determine that **none exists**).
2. Improve the feasible solution to an **optimal** solution.