Applications of Linear Programming

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Lecture 3

Pricing interpretation

Let us go back to our usual manufacturing LP problem. For the sake of illustration, we drop the 3rd constraint, and consider the items as blocks of wood and cans of paint (instead of shop hours).

Manufacturer	Market
$$\begin{array}{rcrcrl} \text{Max } 3x_1 & + & 2x_2 & & & \\ x_1 & + & x_2 & \le & 80 & \text{[wood]} \\ 2x_1 & + & x_2 & \le & 100 & \text{[paint]} \\ & & x_1, x_2 & \ge & 0 & \end{array}$$	Prices: y_1 = price (in \$) of one block of wood y_2 = price (in \$) of one can of paint

Manufacturer owns 80 blocks of wood and 100 cans of paint. He can sell his stock at market prices or buy additional stock at market prices. He can also produce and sell goods (toys) using the available stock.

What is his best strategy (assuming everything produced can be sold)?

Pricing interpretation

- Selling stock generates a profit of $80y_1 + 100y_2$.
- If the cost (in market prices) of producing x_1 toy soldiers is strictly less than the sale price, i.e. if

$$y_1 + 2y_2 < 3$$

- then there is **no limit on the profit** of manufacturer. He can generate arbitrarily large profit by buying additional stock to produce toy soldiers in arbitrary amounts. **Why**?

The production cost of one toy soldier is $y_1 + 2y_2 \Rightarrow$ in case of x_1 toy soldier the cost is $(y_1 + 2y_2)x_1$.

Suppose that $y_1 + 2y_2 = 2.9\$$, and the selling price is $3\$ \Rightarrow$ the profit by selling one toy soldier is $0.1\$ \Rightarrow$ the profit by selling x_1 is $0.1x_1$, that can be arbitrarily large.

The situation is similar in the case of trains.

Pricing interpretation

Market (the competition) will "not allow" the manufacturer to make arbitrarily large profit. It will set its prices so that the manufacturer makes as little as possible. The market is thus "solving" the following:

$$\left.\begin{array}{rcrcll}
\text{Min } 80y_1 & + & 100y_2 & & & \\
y_1 & + & 2y_2 & \geq & 3 & \text{[toy soldiers]} \\
y_1 & + & y_2 & \geq & 2 & \text{[toy trains]} \\
 & & y_1, y_2 & \geq & 0 &
\end{array}\right\} \textbf{Dual} \text{ of the manufacturing problem}$$

The **primal-dual pair** is

$$\textbf{Primal} \quad \begin{array}{rcrcr}
\text{Max } 3x_1 & + & 2x_2 & & \\
x_1 & + & x_2 & \leq & 80 \\
2x_1 & + & x_2 & \leq & 100 \\
 & & x_1, x_2 & \geq & 0
\end{array} \qquad
\begin{array}{rcrcr}
\text{Min } 80y_1 & + & 100y_2 & & \\
y_1 & + & 2y_2 & \geq & 3 \\
y_1 & + & y_2 & \geq & 2 \\
 & & y_1, y_2 & \geq & 0
\end{array} \quad \textbf{Dual}$$

Primal-dual pair in general

Primal LP

$$\begin{array}{lrcl} \text{Max} & c^T x & = & z \\ & Ax & \leq & b \\ & x & \geq & 0 \end{array}$$

Dual LP

$$\begin{array}{lrcl} \text{Min} & b^T y & = & w \\ & A^T y & \geq & c \\ & y & \geq & 0 \end{array}$$

The dual (in standard form) can be easily obtained from primal by

- transposing (flipping around the diagonal) the matrix A,
- swapping vectors b and c,
- switching the inequalities to $\geq$, and
- changing max to min.

Primal-dual pair

Proposition *The dual of the dual is the original primal LP*

Proof. Minimize something it is equivalent to maximize its negative, and we can change the direction of the inequalities

Dual LP

$$\sum_{i=1}^{m}(-a_{ij})y_i \le -c_j \qquad j = 1, 2, \ldots n$$

$$y_i \ge 0 \qquad i = 1, 2, \ldots m$$

$$\max \sum_{i=1}^{m}(-b_i)y_i = w$$

Dual of the dual

$$\sum_{j=1}^{n}(-a_{ij})x_j \ge -b_i \qquad i = 1, 2, \ldots m$$

$$x_j \ge 0 \qquad j = 1, 2, \ldots n$$

$$\min \sum_{j=1}^{n} -(c_j)x_j = z$$

which is clearly equivalent to the primal problem

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Weak duality theorem

Theorem. (Weak duality) *If $x = (x_1, \ldots, x_n)$ is feasible solution for the primal and $y = (y_1, \ldots, y_m)$ is feasible solution for the dual then* $c^T x \le b^T y$, i.e.

$$\sum_{j=1}^{n} c_j x_j \le \sum_{i=1}^{m} b_i y_i.$$

It means that any feasible solution of the dual is an upper bound of all feasible solution (contains the primal optimum in particular) of the primal.

Proof. The proof is a simple chain of obvious inequalities:

$$\sum_{j=1}^{n} c_j x_j \le \sum_{j=1}^{n} \left(\sum_{i=1}^{m} y_i a_{ij} \right) x_j = \sum_{i=1}^{m} \left(\sum_{j=1}^{n} x_j a_{ij} \right) y_i \le \sum_{i=1}^{m} b_i y_i,$$

or in matrix form:

$$c^T x \le (A^T y)^T x = (y^T A) x = y^T (Ax) \le y^T b = b^T y$$

$\blacksquare$

Strong duality theorem

Theorem. (Strong duality) *If the primal has an optimal solution $x^* = (x_1, \ldots, x_n)$ then the dual has an optimal solution $y^* = (y_1, \ldots, y_m)$ such that $c^T x = b^T y$, i.e.*

$$\sum_{j=1}^{n} c_j x_j = \sum_{i=1}^{m} b_i y_i.$$

Furthermore the following equalities hold:

$$y^T(b - Ax) = 0 \text{ és } x^T(A^T y - c) = 0.$$

Simply, if ith **constraint inequality is not sharp** *($=$ no equality) in the primal optimum, then* **the corresponding dual y_i variable is** 0*.*

Backwards, if primal x_i variable is strictly positive in the optimum, then the corresponding dual constraint is sharp ($=$ holds).

This is called **complementary slackness**.

Strong duality theorem

Proof of the second part (complementary slackness):

$$0 \leq y^T(b - Ax) = y^T b - y^T Ax = b^T y - (A^T y)^T x \leq b^T y - c^T x = 0,$$

and

$$0 \leq x^T(A^T y - c) = (y^T A - c^T)x = y^T(Ax) - c^T x \leq y^T b - c^T x = b^T y - c^T x = 0.$$

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Primal-dual possible cases

- Strong duality theorem $\Rightarrow$ if the primal problem has an optimal solution, the dual problem has one also and there is *no duality gap*.
- But what if the primal problem does not have an optimal solution? E.g. it is unbounded.
- The unboundedness of the primal $+$ weak duality theorem $\Rightarrow$ dual problem must be infeasible.
- Similarly, if the dual problem is unbounded $\Rightarrow$ the primal problem must be infeasible.

		primal	
dual	infeasible	feasible bounded	unbounded
infeasible	$\checkmark$	$\times$	$\checkmark$
feasible bounded	$\times$	$\checkmark$	$\times$
unbounded	$\checkmark$	$\times$	$\times$

$\checkmark$ possible

$\times$ impossible

General LP

- If LP contains equalities or unrestricted variables, these can be also handled with ease.
- In particular, **equality** constraint corresponds to an **unrestricted variable**, and vice-versa
- Why? Suppose that $x_1 + x_2 = 80$

$$3x_1 + 2x_2 \leq 5x_1 + 2x_2 = (-1)\underbrace{(x_1 + x_2)}_{=80} + 3\underbrace{(2x_1 + x_2)}_{\leq 100} \leq$$

$$\leq -80 + 3 \times 100 = 220\$$$

thus y_1 is unrestricted (now it is -1).

General LP

- Conversely, suppose that x_1 has no sign restriction.
- Then then we **could not conclude** that

$$3x_1 + 2x_2 \leq 4x_1 + 2x_2$$

  holds for all feasible solutions (e.g. if $x_1 = -1$; $x_2 \geq 0$ holds)
- In general $3x_1 \leq (y_1 + 2y_2)x_1$ $[*]$, thus set $y_1 + 2y_2$ to its maximal value 3, $[*]$ still holds for unrestricted x_1
- Similarly true that
  - a primal "$\geq$" constraint corresponds to an non-positive variable, and vice-versa.

General LP

$\geq$ constraint corresponds to an **non-positive** variable, and **vice-versa**.

Primal (Max)	Dual (Min)
i-th constraint $\leq$	variable $y_i \geq 0$
i-th constraint $\geq$	variable $y_i \leq 0$
i-th constraint $=$	variable y_i unrestricted
$x_i \geq 0$	i-th constraint $\geq$
$x_i \leq 0$	i-th constraint $\leq$
x_i unrestricted	i-th constraint $=$

$$
\begin{array}{lrcrcrcl}
\text{Max} & 3x_1 & + & 2x_2 & + & x_3 & & \\
& x_1 & + & x_2 & + & \frac{1}{2}x_3 & \leq & 80 \\
& 2x_1 & + & x_2 & + & x_3 & = & 100 \\
& x_1 & & & + & x_3 & \geq & 40 \\
& & & & & x_1 & \multicolumn{2}{l}{\text{unrestricted}} \\
& & & & & x_2 & \leq & 0 \\
& & & & & x_3 & \geq & 0
\end{array}
$$

Primal

$$
\begin{array}{lrcrcrcl}
\text{Min} & 80y_1 & + & 100x_2 & + & 40x_3 & & \\
& y_1 & + & 2y_2 & + & y_3 & = & 3 \\
& y_1 & + & y_2 & & & \leq & 2 \\
& \frac{1}{2}y_1 & + & y_2 & + & y_3 & \geq & 1 \\
& & & & & y_1 & \geq & 0 \\
& & & & & y_2 & \multicolumn{2}{l}{\text{unrestricted}} \\
& & & & & y_3 & \leq & 0
\end{array}
$$

Dual

LP solvability

Inconsistency: an m element system of equations and inequalities

$$\sum_{j=1}^{n} a_{ij}x_j \leq b_i \qquad i \in I$$

$$\sum_{j=1}^{n} a_{ij}x_j = b_i \qquad i \in E$$

is inconsistent, if there are $y_1, y_2, \ldots, y_m$ real numbers such that

$$\sum_{i=1}^{m} a_{ij}y_i = 0 \qquad j = 1, 2, \ldots, n$$

$$\sum_{i=1}^{m} b_i y_i < 0$$

$$y_i \geq 0 \qquad i \in I$$

is satisfied.

LP solvability

Tucker's impossibility theorem for system of equations and inequalities.

System of equations and inequalities **is unsolvable if and only if it is inconsistent**

Problem 1

In a vegetable shop the following number of workers are needed from Monday to Sunday, respectively): M:27 T:24 W:23 T:20 F:25 Sa:27 Su:28 (this is necessary to run the business, but we are flexible, means that a bit more employees in each day is not a big problem). Further requirement that an employee must work in 5 consecutive days, e.g. from Monday to Friday, from Tuesday to Saturday, etc. The task of the ownership to satisfy all conditions with a minimum number of employees (therefore minimize his wage cost).

Problem 2

The Bloomington Brewery Inc. produces pilsner type beers and ale type beers. The price of the pilsner type is 40 euros per barrel, while the price of the ale type is 30 euros per barrel. To produce one barrel pilsner they need 5 kg barely malt and 2 kg hop. On the other hand, to produce one barrel ale 3 kg barely malt and 1 kg hop are needed. In a given day, they have 60 kg barley malt and 25 kg hop in storage. The task is to determine the production numbers in order to maximize the expected income.

Problem 3

The Dorian Cars company distribute luxury cars and lorries/trucks. The management observed that the costumers are mostly men and women with high income. To reach this group the company is going to start an advertising campaign and buy 30 seconds advertising space during two kind of TV broadcasts: comedies and football games. Estimations show that each commercial during a comedy is watched by 7 million women (with high salary) and 2 million men, while in case of a football game the numbers are 2 millions and 12 millions, respectively. The cost of a commercial is 5,000 dollars in case of a comedy and 10,000 dollars in case of a football game. The ownership wants that the commercials would be seen by at least 28 million women and 24 million men reached by the minimum commercial cost.