Applications of Linear Programming

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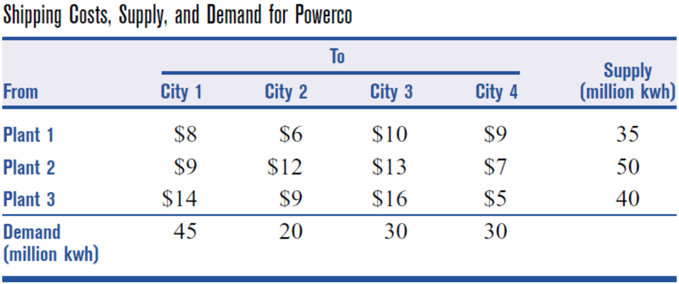
Lecture 7
based on the book Operation Research by Wayne L. Winston

- We discuss three special types of linear programming problems:
  1. **transportation**
  2. **assignment**
  3. **transshipment**
- Each of these can be solved by the simplex algorithm, but **specialized algorithms** for each type of problem are **much more efficient.**

Example 1

PowerCo has three electric power plants that supply the needs of four cities. Each power plant can supply the following numbers of kilowatt-hours (kwh) of electricity: plant 1 - 35 million; plant 2 - 50 million; plant 3 - 40 million. The peak power demands in these cities, which occur at the same time (2 P.M.), are as follows (in kwh): city 1 - 45 million; city 2 - 20 million; city 3 - 30 million; city 4 - 30 million. The costs of sending 1 million kwh of electricity from plant to city depend on the distance the electricity must travel. Formulate an LP to minimize the cost of meeting each city's peak power demand.



Shipping Costs, Supply, and Demand for Powerco

	To				
From	City 1	City 2	City 3	City 4	Supply (million kwh)
Plant 1	\$8	\$6	\$10	\$9	35
Plant 2	\$9	\$12	\$13	\$7	50
Plant 3	\$14	\$9	\$16	\$5	40
Demand (million kwh)	45	20	30	30	

Example 1 - solution

PowerCo must determine how much power is sent from each plant to each city, we define (for $i = 1, 2, 3$ and $j = 1, 2, 3, 4$)

x_{ij} = number of (million) kwh produced at plant i and sent to city j

In terms of these variables, the total cost of supplying the peak power demands to cities 1–4 may be written as

$$\begin{aligned} &8x_{11} + 6x_{12} + 10x_{13} + 9x_{14} \text{ (Cost of shipping power from plant 1)} \\ &+9x_{21} + 12x_{22} + 13x_{23} + 7x_{24} \text{ (Cost of shipping power from plant 2)} \\ &+14x_{31} + 9x_{32} + 16x_{33} + 5x_{34} \text{ (Cost of shipping power from plant 3)} \end{aligned}$$

Example 1 - solution

The LP formulation of PowerCo's problem contains the following three supply constraints:

$x_{11} + x_{12} + x_{13} + x_{14} \leq 35$ (Plant 1 supply constraint)

$x_{21} + x_{22} + x_{23} + x_{24} \leq 50$ (Plant 2 supply constraint)

$x_{31} + x_{32} + x_{33} + x_{34} \leq 40$ (Plant 3 supply constraint)

PowerCo must satisfy the following four demand constraints:

$x_{11} + x_{21} + x_{31} \geq 45$ (City 1 demand constraint)

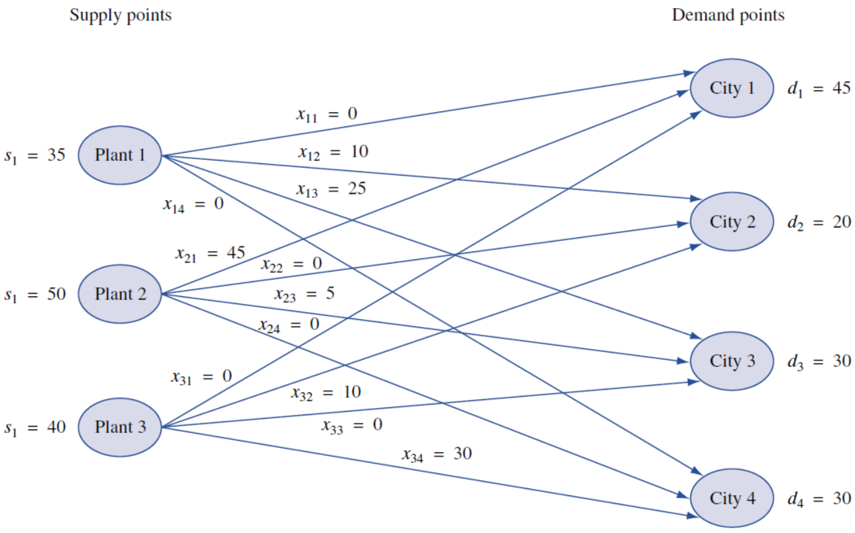
$x_{12} + x_{22} + x_{32} \geq 20$ (City 2 demand constraint)

$x_{13} + x_{23} + x_{33} \geq 30$ (City 3 demand constraint)

$x_{14} + x_{24} + x_{34} \geq 30$ (City 4 demand constraint)

Because all the x_{ij} must be nonnegative, we add the sign restrictions $x_{ij} \geq 0$ $(i = 1, 2, 3; j = 1, 2, 3, 4)$.

Example 1 - graphical representation



General description

In general, a transportation problem is specified by the following information:

1. A set of m *supply points* from which a good is shipped. Supply point i can supply at most s_i units. In the Powerco example, $m = 3$, $s_1 = 35$, $s_2 = 50$, and $s_3 = 40$.
2. A set of n *demand points* to which the good is shipped. Demand point j must receive at least d_j units of the shipped good. In the Powerco example, $n = 4$, $d_1 = 45$, $d_2 = 20$, $d_3 = 30$, and $d_4 = 30$.
3. Each unit produced at supply point i and shipped to demand point j incurs a *variable cost* of c_{ij}. In the Powerco example, $c_{12} = 6$.

Let

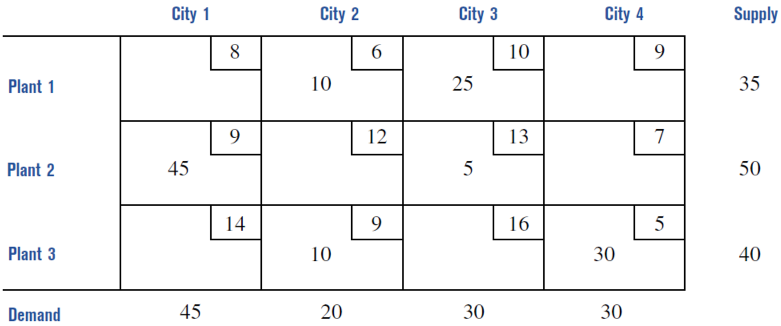
$$x_{ij} = \text{number of units shipped from supply point } i \text{ to demand point } j$$

then the general formulation of a transportation problem is

$$\begin{aligned}
&\min \sum_{i=1}^{i=m} \sum_{j=1}^{j=n} c_{ij}x_{ij} \\
&\text{s.t.} \quad \sum_{j=1}^{j=n} x_{ij} \le s_i \quad (i = 1, 2, \ldots, m) \qquad \text{(Supply constraints)} \\
&\qquad\quad \sum_{i=1}^{i=m} x_{ij} \ge d_j \quad (j = 1, 2, \ldots, n) \qquad \text{(Demand constraints)} \\
&x_{ij} \ge 0 \quad (i = 1, 2, \ldots, m; j = 1, 2, \ldots, n)
\end{aligned}$$

Transportation tableau

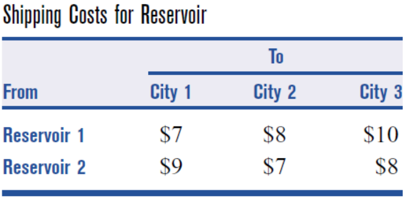
A transportation problem is specified by the supply, the demand, and the shipping costs, so the relevant data can be summarized in a **transportation tableau**.The square, or cell, in row i and column j of a transportation tableau corresponds to the variable x_{ij}



	City 1	City 2	City 3	City 4	Supply
Plant 1	8	6 10	10 25	9	35
Plant 2	9 45	12	13 5	7	50
Plant 3	14	9 10	16	5 30	40
Demand	45	20	30	30	

Example 2

Two reservoirs are available to supply the water needs of three cities. Each reservoir can supply up to 50 million gallons of water per day. Each city would like to receive 40 million gallons per day. For each million gallons per day of unmet demand, there is a penalty. At city 1, the penalty is \$20; at city 2, the penalty is \$22; and at city 3, the penalty is \$23. The cost of transporting 1 million gallons of water from each reservoir to each city is shown in the table.Formulate a balanced transportation problem that can be used to minimize the sum of shortage and transport costs.



Shipping Costs for Reservoir

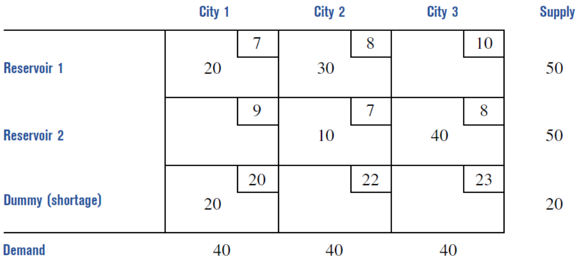
	To		
From	City 1	City 2	City 3
Reservoir 1	\$7	\$8	\$10
Reservoir 2	\$9	\$7	\$8

Example 2 - solution

In this problem,

$$\text{Daily supply} = 50 + 50 = 100 \text{ million gallons per day}$$
$$\text{Daily demand} = 40 + 40 + 40 = 120 \text{ million gallons per day}$$

To balance the problem, we add a dummy (or shortage) supply point having a supply of $120 - 100 = 20$ million gallons per day. The cost of shipping 1 million gallons from the dummy supply point to a city is just the shortage cost per million gallons for that city. Table shows the balanced transportation problem and its optimal solution.



	City 1	City 2	City 3	Supply
Reservoir 1	7 20	8 30	10	50
Reservoir 2	9	7 10	8 40	50
Dummy (shortage)	20 20	22	23	20
Demand	40	40	40	

Twenty million gallons per day of city 1's demand will be unsatisfied.

Transportation simplex method

Three methods that can be used to **find a basic feasible solution** for a balanced transportation problem are

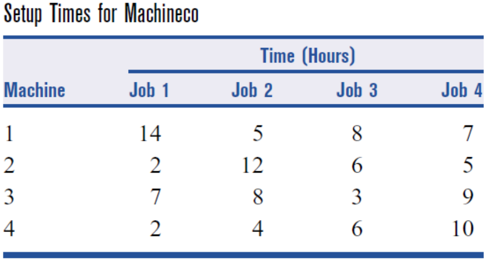
1. northwest corner method
2. minimum-cost method
3. Vogel’s method

We will not discuss them here, but good to know that they work efficiently.

Although the transportation simplex appears to be very efficient, there is a certain class of transportation problems, called **assignment problems**, for which the transportation simplex is often very inefficient.

Example 1

MachineCo has four machines and four jobs to be completed. Each machine must be assigned to complete one job. The time required to set up each machine for completing each job is shown in the table.



Setup Times for Machineco

	Time (Hours)			
Machine	**Job 1**	**Job 2**	**Job 3**	**Job 4**
1	14	5	8	7
2	2	12	6	5
3	7	8	3	9
4	2	4	6	10

MachineCo wants to minimize the total setup time needed to complete the four jobs. Use linear programming to solve this problem.

Example 1 - solution

Machineco must determine which machine should be assigned to each job. We define (for $i, j = 1, 2, 3, 4$)

$$x_{ij} = 1 \text{ if machine } i \text{ is assigned to meet the demands of job } j$$
$$x_{ij} = 0 \text{ if machine } i \text{ is not assigned to meet the demands of job } j$$

Then Machineco's problem may be formulated as

$$\begin{aligned}
\min z = {} & 14x_{11} + 5x_{12} + 8x_{13} + 7x_{14} + 2x_{21} + 12x_{22} + 6x_{23} + 5x_{24} \\
& + 7x_{31} + 8x_{32} + 3x_{33} + 9x_{34} + 2x_{41} + 4x_{42} + 6x_{43} + 10x_{44} \\
\text{s.t.} \quad & x_{11} + x_{12} + x_{13} + x_{14} = 1 \qquad \text{(Machine constraints)} \\
& x_{21} + x_{22} + x_{23} + x_{24} = 1 \\
& x_{31} + x_{32} + x_{33} + x_{34} = 1 \\
& x_{41} + x_{42} + x_{43} + x_{44} = 1 \\
& x_{11} + x_{21} + x_{31} + x_{41} = 1 \qquad \text{(Job constraints)} \\
& x_{12} + x_{22} + x_{32} + x_{42} = 1 \\
& x_{13} + x_{23} + x_{33} + x_{43} = 1 \\
& x_{14} + x_{24} + x_{34} + x_{44} = 1 \\
& x_{ij} = 0 \quad \text{or} \quad x_{ij} = 1
\end{aligned}$$

Example 1 - solution

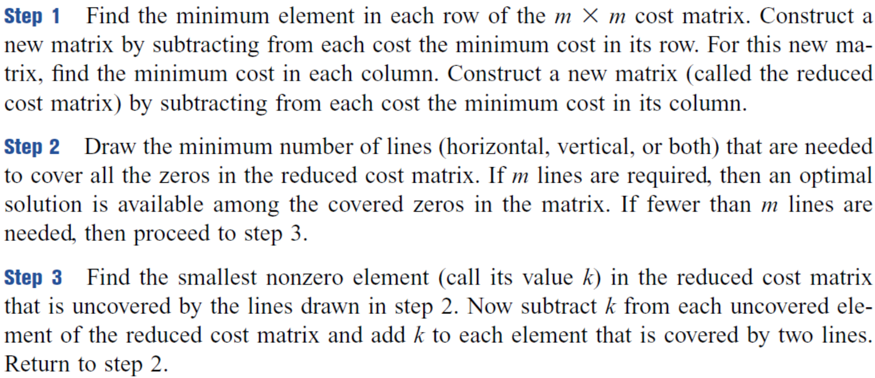
The first four constraints ensure that each machine is assigned to a job, and the last four ensure that each job is completed. If $x_{ij} = 1$, then the objective function will pick up the time required to set up machine i for job j; if $x_{ij} = 0$, then the objective function will not pick up the time required.

Ignoring for the moment the $x_{ij} = 0$ or $x_{ij} = 1$ restrictions, we see that MachineCo faces a **balanced transportation problem** in which each supply point has a supply of 1 and each demand point has a demand of 1.

In general, an assignment problem is a balanced transportation problem in which all supplies and demands are equal to 1.

Solving the assignment problem - The Hungarian method

Thus, an assignment problem is characterized by knowledge of the cost of assigning each supply point to each demand point. The assignment problem's matrix of costs is its **cost matrix**.



Step 1 Find the minimum element in each row of the $m \times m$ cost matrix. Construct a new matrix by subtracting from each cost the minimum cost in its row. For this new matrix, find the minimum cost in each column. Construct a new matrix (called the reduced cost matrix) by subtracting from each cost the minimum cost in its column.

Step 2 Draw the minimum number of lines (horizontal, vertical, or both) that are needed to cover all the zeros in the reduced cost matrix. If m lines are required, then an optimal solution is available among the covered zeros in the matrix. If fewer than m lines are needed, then proceed to step 3.

Step 3 Find the smallest nonzero element (call its value k) in the reduced cost matrix that is uncovered by the lines drawn in step 2. Now subtract k from each uncovered element of the reduced cost matrix and add k to each element that is covered by two lines. Return to step 2.

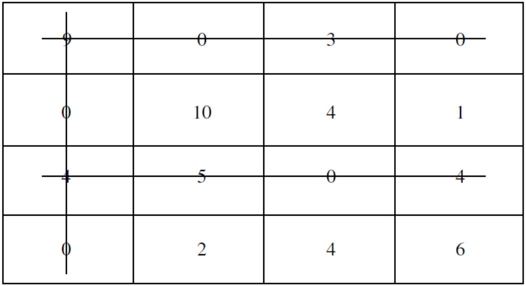
The Hungarian method

				Row Minimum
14	5	8	7	5
2	12	6	5	2
7	8	3	9	3
2	4	6	10	2

9	0	3	2
0	10	4	3
4	5	0	6
0	2	4	8
Column Minimum	0	0	2

The Hungarian method

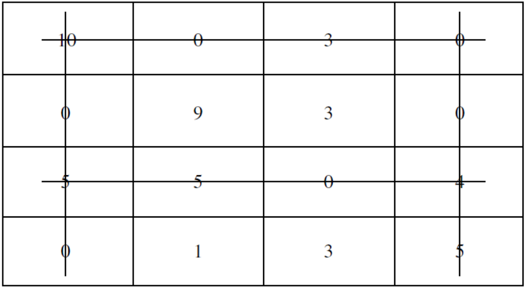
Cost matrix after column minimums are subtracted:



9	0	3	0
0	10	4	1
4	5	0	4
0	2	4	6

Step 3: The smallest uncovered element equals 1, so we now subtract 1 from each uncovered element in the reduced cost matrix and add 1 to each twice-covered element. The resulting matrix is in the next slide.

The Hungarian method



10	0	3	0
0	9	3	0
5	5	0	4
0	1	3	5

Four lines are now required to cover all the zeros. Thus, an optimal solution is available. To find an optimal assignment, observe that the only covered 0 in column 3 is x_{33}, so we must have $x_{33} = 1$. Also, the only available covered zero in column 2 is x_{12}, so we set $x_{12} = 1$ and observe that neither row 1 nor column 2 can be used again. Now the only available covered zero in column 4 is x_{24}. Thus, we choose $x_{24} = 1$ (which now excludes both row 2 and column 4 from further use). Finally, we choose $x_{41} = 1$.

A transportation problem allows only shipments that go directly from a supply point to a demand point.

In many situations, shipments are allowed between supply points or between demand points. Sometimes there may also be points (called transshipment points) through which goods can be transshipped on their journey from a supply point to a demand point. Shipping problems with any or all of these characteristics are **transshipment problems**.

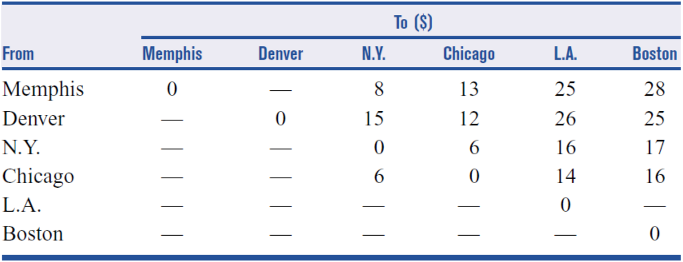
Fortunately, the optimal solution to a transshipment problem can be found by solving a transportation problem.

- We define a **supply point** to be a point that can send goods to another point but cannot receive goods from any other point
- Similarly, a **demand point** is a point that can receive goods from other points but cannot send goods to any other point.
- A **transshipment point** is a point that can both receive goods from other points and send goods to other points.

Example 1

WidgetCo manufactures widgets at two factories, one in Memphis and one in Denver. The Memphis factory can produce as many as 150 widgets per day, and the Denver factory can produce as many as 200 widgets per day. Widgets are shipped by air to customers in Los Angeles and Boston. The customers in each city require 130 widgets per day. Because of the deregulation of airfares, WidgetCo believes that it may be cheaper to first fly some widgets to New York or Chicago and then fly them to their final destinations. The costs of flying a widget are shown in Table. WidgetCo wants to minimize the total cost of shipping the required widgets to its customers.

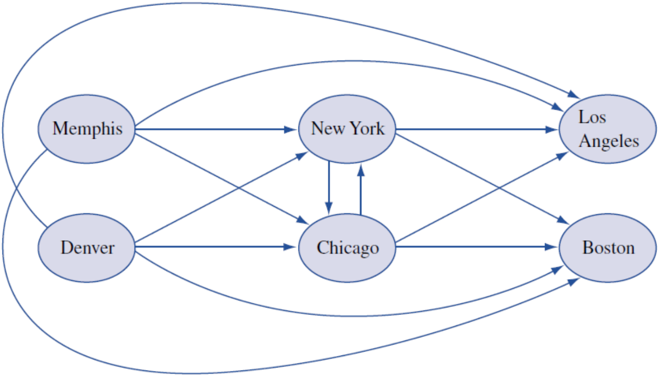
Shipping Costs for Transshipments



	To ($)					
From	**Memphis**	**Denver**	**N.Y.**	**Chicago**	**L.A.**	**Boston**
Memphis	0	—	8	13	25	28
Denver	—	0	15	12	26	25
N.Y.	—	—	0	6	16	17
Chicago	—	—	6	0	14	16
L.A.	—	—	—	—	0	—
Boston	—	—	—	—	—	0

Example 1 - solution

- Memphis and Denver are supply points, with supplies of 150 and 200 widgets per day, respectively
- New York and Chicago are transshipment points
- Los Angeles and Boston are demand points, each with a demand of 130 widgets per day



Example 1 - solution

- The optimal solution to a transshipment problem can be found by solving a transportation problem
- Given a transshipment problem, we create a balanced transportation problem by the following procedure (assume that total supply exceeds total demand):
    1. If necessary, add a dummy demand point to balance the problem
    2. Construct a transportation tableau: Because $s = (\text{total supply}) = 150 + 200 = 350$ and $d = (\text{total demand}) = 130 + 130 = 260$, the dummy demand point has a demand of $350 - 260 = 90$. The other supplies and demands in the transportation tableau are obtained by adding $s = 350$ to each transshipment point's supply and demand.

Example 1 - solution

The net outflow from each city and the tableu:

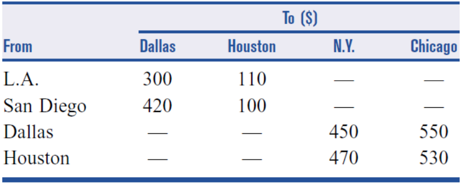
Memphis: 130 + 20 = 150
Denver: 130 + 70 = 200
N.Y.: 220 + 130 − 130 − 220 = 0
Chicago: 350 − 350 = 0
L.A.: −130
Boston: −130
Dummy: −20 − 70 = −90

	N.Y.	**Chicago**	**L.A.**	**Boston**	**Dummy**
Memphis	8 130	13	25	28	0 20
Denver	15	12	26	25 130	0 70
N.Y.	0 220	6	16 130	17	0
Chicago	6	0 350	14	16	0
Demand	350	350	130	130	90

- Negative outflow means inflow.
- Transshipment points has a net outflow 0.
- In interpreting the solution to the transportation problem created from a transshipment problem, we simply ignore the shipments to the dummy and from a point to itself

Exercise 1

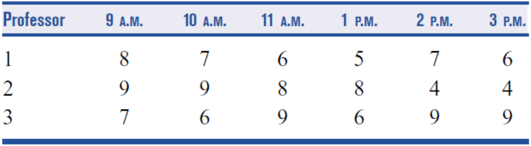
OilCo has oil fields in San Diego and Los Angeles. The San Diego field can produce 500,000 barrels per day, and the Los Angeles field can produce 400,000 barrels per day. Oil is sent from the fields to a refinery, either in Dallas or in Houston (assume that each refinery has unlimited capacity). It costs \$700 to refine 100,000 barrels of oil at Dallas and \$900 at Houston. Refined oil is shipped to customers in Chicago and New York. Chicago customers require 400,000 barrels per day of refined oil; New York customers require 300,000. The costs of shipping 100,000 barrels of oil (refined or unrefined) between cities are given in the table. Formulate a balanced transportation model of this situation.



	To ($)			
From	**Dallas**	**Houston**	**N.Y.**	**Chicago**
L.A.	300	110	—	—
San Diego	420	100	—	—
Dallas	—	—	450	550
Houston	—	—	470	530

Exercise 2

Three professors must be assigned to teach six sections of finance. Each professor must teach two sections of finance, and each has ranked the six time periods during which finance is taught, as shown in the table. A ranking of 10 means that the professor wants to teach that time, and a ranking of 1 means that he or she does not want to teach at that time. Determine an assignment of professors to sections that will maximize the total satisfaction of the professors.



Professor	9 A.M.	10 A.M.	11 A.M.	1 P.M.	2 P.M.	3 P.M.
1	8	7	6	5	7	6
2	9	9	8	8	4	4
3	7	6	9	6	9	9