ON THE EDGE-BANDWIDTH OF GRAPH PRODUCTS

JÓZSEF BALOGH, DHRUV MUBAYI, AND ANDRÁS PLUHÁR

Abstract. The edge-bandwidth of a graph G is the bandwidth of the line graph of G. We show asymptotically tight bounds on the edge-bandwidth of two dimensional grids and tori, the product of two cliques and the n-dimensional hypercube.

1. Introduction

Let $G = (V(G), E(G))$ be a simple graph with n vertices. A *labelling* η is a bijection of $V(G)$ to $\{1, \dots, n\}$. The *bandwidth* of η is

$$B(\eta, G) = \max\{|\eta(u) - \eta(v)| : uv \in E(G)\}.$$

The *bandwidth* $B(G)$ of G is

$$B(G) := \min_{\eta}\{B(\eta, G)\}.$$

The notion first came up in the seminal paper of Harper [7] in which the bandwidth of the n-dimensional hypercube was given. It turns out that the determination or computation of the bandwidth of graphs is hard (in fact, it is NP-hard [15]); for a good survey, see [4], [5] or [13].

The edge-bandwidth was introduced by Hwang and Lagarias in [10]. Here the edges are labelled instead of the vertices, and the *bandwidth* of an edge-labelling η of a graph G is

$$B'(\eta, G) := \max\{|\eta(uv) - \eta(vw)| : uv, vw \in E(G)\}.$$

The edge-bandwidth of a graph G is

$$B'(G) := \min_{\eta}\{B'(\eta, G)\}.$$

Of course $B'(G) = B(L(G))$, where $L(G)$ is the line graph of G, see [11]. The *Cartesian product* of graphs G and H is denoted by $G \oplus H$, with $V(G \oplus H) = \{(u, v)|\ u \in V(G),\ v \in V(H)\}$ and $E(G \oplus H) = \{\langle (u_1, v_1), (u_2, v_2)\rangle |\ u_1 = u_2, (v_1, v_2) \in E(H)$ or $(u_1, u_2) \in E(G),\ v_1 = v_2\}$. The n-th fold product $G \oplus G \oplus \dots \oplus G$ is denoted

1991 *Mathematics Subject Classification.* 05C78.

Key words and phrases. Bandwidth, edge-bandwidth, line graph, hypercube, grid, isoperimetric problems.

The first authors research is partially supported by NSF grant DMS-0302804.

The second authors research is partially supported by NSF grant DMS-0400812.

The third author was partially supported by OTKA grant T34475.

G^n. In this paper we shall give estimates for edge-bandwidth of four types of graph products, $P_n \oplus P_n$, where P_n denotes the path with n vertices, $C_n \oplus C_n$, where C_n is the cycle on n vertices, $K_n \oplus K_n$, where K_n is the clique on n vertices, and $P_2^n = K_2^n$, the n-dimensional hypercube. The bandwidths of $P_n \oplus P_n$, $C_n \oplus C_n$ and $K_n \oplus K_n$ are well studied, the first one is n, the second is $2n-1$, while the third one is $\lfloor (n^2+n-1)/2 \rfloor$, see [6, 14, 1]. Nevertheless, only the trivial lower bounds n on $B'(P_n \oplus P_n)$ and $n^2(n-1)/3 \le B'(K_n \oplus K_n)$ are known on the edge-bandwidth of those. (One can readily get those by Proposition 5, in the next section.) Note that it is easy to see that $B'(P_n) = 1, B'(C_n) = 2$, and in [11] it was proved that $B'(K_n) = \lfloor n^2/4 \rfloor + \lceil n/2 \rceil - 2$ and $B'(K_{n,n}) = \binom{n+1}{2} - 1$ where $K_{n,n}$ denotes the complete bipartite graph. Our results are the following.

Theorem 1. *Let $n \ge 2$. Then*

$$2n - \sqrt{n} - 1 \le B'(P_n \oplus P_n) \le 2n - 1 \tag{1}$$

and

$$4n - 2\sqrt{2n} - 1 \le B'(C_n \oplus C_n) \le 4n. \tag{2}$$

We also obtain asymptotically tight bounds on the edge-bandwidth of the product of two equal cliques.

Theorem 2.

$$\frac{3n^3}{8} - \frac{n^2}{16} - \frac{7n}{16} + \frac{3}{16} \le B'(K_n \oplus K_n) \le \frac{3n^3}{8} + \frac{19n^2}{8}. \tag{3}$$

The third family of graphs have been studied extensively earlier. Recall that P_2^n is the n-dimensional hypercube, that is the vertices of P_2^n are the 0-1 sequences of length n, and there is an edge between the vertices x and y iff their Hamming distance is one. Bezrukov, Grünwald and Weber [2] showed that

$$2^{n-1} + 2^{n-2} \le B'(P_2^n) \le 2\left\lceil \frac{n}{2} \right\rceil \binom{n}{\lfloor \frac{n}{2} \rfloor} - 1.$$

An improved lower bound on $B'(P_2^n)$ was proved by Calamoneri, Massini and Vrťo [3], namely that

$$\frac{n}{4}\binom{n}{\lfloor \frac{n}{2} \rfloor} \le B'(P_2^n).$$

Our final result establishes the right asymptotical growth of $B'(P_2^n)$:

Theorem 3.

$$B'(P_2^n) = \left(\frac{n}{2} + o(n)\right)\binom{n}{\lceil \frac{n}{2} \rceil}. \tag{4}$$

2. General Bounds

The standard techniques for obtaining lower bounds on bandwidth apply isoperimetric inequalities. In the literature many vertex and edge isoperimetric problems were considered, in particular on the square grid and the hypercube. Given a graph G, for an $S \subset V(G)$ let

$$\partial(S) = \{v \in V(G) \setminus S |\, (u,\, v) \in E(G),\, u \in S\}.$$

A typical (vertex) isoperimetric question is that for a given graph G of order n, and for a fixed integer k what is

$$L_k(G) := \min_{|S|=k,\ S \subset V(G)} |\partial(S)|?$$

As Proposition 4 states, the value of $\max_k L_k(G)$ is a lower bound for $B(G)$. It is not hard to see that this bound is sharp if the extremal structures for different k's, achieving the isoperimetric bound, can be positioned in G to be built a "nested" sequence of sets, more precisely a sequence $\{S_k\}_{k=1}^n \subset V(G)$ can be build that for all $i < j$, $S_i \subset S_j$ and either $S_i \cup \partial(S_i) \subset S_j$ or $S_j \subset S_i \cup \partial(S_i)$. See [8] for more details.

In our cases, Proposition 4 does not even give asymptotically sharp bounds, but surprisingly the iterated version of it, Proposition 6 does. In particular, we remark that the vertex isoperimetric number of $L(P_n \oplus P_n) \leq n + 1$, but the bandwidth is around $2n$.

Proposition 4. [7] *Let G be a graph and k be an integer, $0 \leq k \leq |V(G)|$. Then*

$$B(G) \geq \min_{S,\, |S|=k} \max\{|\partial(S)|, |\partial(V - S)|\}.$$

The *length* of a P_n is $n - 1$. The *distance* of two vertices in a graph G is the length of the shortest path between them, and the *diameter*, $\mathrm{diam}(G)$, of a graph G is the maximum distance between its vertices.

Proposition 5. [5] *Let G be a graph. Then*

$$B(G) \geq \frac{|V(G)| - 1}{\mathrm{diam(G)}}.$$

Since we need an extension of these results, we give the outline of their proofs. Fix a labelling of G, and let S be the set of vertices labelled by the numbers $\{1, \ldots, k\}$. Now the largest number appearing on the vertices in $\partial(S)$ is at least $k + |\partial(S)|$, which gives a (absolute) difference at least $|\partial(S)|$ with the label of some vertex of S; that is $B(G) \geq |\partial(S)|$. Using the same estimate for $V - S$ and taking the optimal labelling, Proposition 4 follows. For Proposition 5 consider a shortest path connecting the vertices labelled by 1 and $|V(G)|$. The average (absolute) difference between the labels of neighboring vertices is at least $(|V(G)| - 1)/\mathrm{diam}(G)$, hence the largest difference is at least this much.

Similarly to Proposition 4, one may consider not only the set $\partial(S)$, but

$$\partial^\ell(S) := \partial(\partial^{\ell-1}(S)) - \cup_{i=1}^{\ell}\partial^{i-1}(S),$$

where $\partial^0(S) := S$. Let $\sigma^\ell(S) := \cup_{i=1}^{\ell}\partial^i(S)$, shortly the *$\ell$-shadow of S*. Note that every vertex of $\sigma^\ell(S)$ is connected to S by a path of length at most ℓ. As before, fix a labelling of G, and let S be the set of vertices labelled by the numbers $\{1,\ldots,k\}$. The biggest label in $\sigma^\ell(S)$ is at least $k+|\sigma^\ell(S)|$, say this appears on vertex $y \in \sigma^\ell(S)$. Consider now a shortest path connecting an arbitrary vertex x of S and vertex y. The average (absolute) difference between the labels of neighboring vertices is at least $|\sigma^\ell(S)|/\ell$, hence the largest difference is at least this much. This yields the following result.

Proposition 6. *Let G be a graph and k an integer, $0 \le k \le |V(G)|$. Then we have*

$$B(G) \ge \min_{S,\,|S|=k}\ \max_{1\le \ell\le n} \frac{|\sigma^\ell(S)|}{\ell}. \tag{5}$$

3. The proof of Theorem 1

3.1. **The case of grids.** We need to prove that $2n-\sqrt{n}-1 \le B'(P_n\oplus P_n) \le 2n-1$. Here we have two candidates for optimal labellings that are different from each other, which perhaps makes finding $B'(P_n\oplus P_n)$ harder, see Figure 1. The vertices of $P_n\oplus P_n$ are labelled with (i,j) where $1 \le i,j \le n$.

Labelling 1. Let

$$\eta(\langle (i,j),(i,j+1)\rangle) := (i-1)(2n-1)+j$$

and

$$\eta(\langle (i,j),(i+1,j)\rangle) := i(2n-1)+j-n.$$

Labelling 2. We consider the vertices of the grid as the elements of an $n\times n$ matrix. For edges below the diagonal $(1,n)-(n,1)$ let

$$\eta(\langle (i,j),(i,j+1)\rangle) := (i+j-2)(i+j-1)+2i-1$$

and

$$\eta(\langle (i,j),(i+1,j)\rangle) := (i+j-2)(i+j-1)+2i.$$

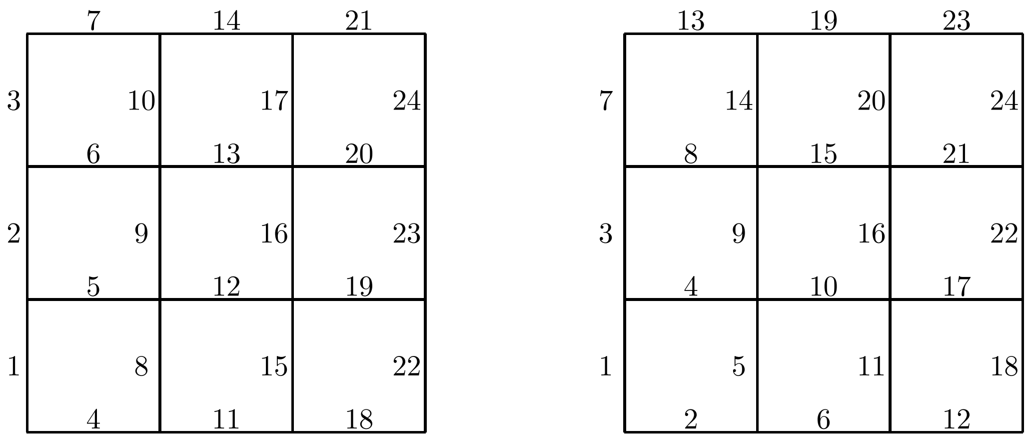
Otherwise we extend the labels in antisymmetric way:

$$\eta(\langle (i,j),(i,j+1)\rangle) := 2n^2+1-2n-\eta(\langle n+1-i,n+1-j\rangle,\langle n+1-i,n-j\rangle)$$

and

$$\eta(\langle (i,j),(i+1,j)\rangle) := 2n^2+1-2n-\eta(\langle n+1-i,n+1-j\rangle,\langle n-i,n+1-j\rangle).$$

It is not hard to check that the bandwidth of both labellings is $2n-1$.



Labelling 1 for $n = 4$ *Labelling 2 for $n = 4$*

Figure 1

For the proof of the lower bound, we need the following Lemma.

Lemma 7. *(i) Let G be the tree consisting of a path of length $n-1$ together with $n-1$ additional edges incident with all but the first vertex of the path. (So $|V(G)| = 2n-1$ and $|E(G)| = 2n-2$.) Let D be a nonempty set of edges in G. Suppose that $|E(G) - D| \geq 2r - 1$ for some $r > 0$. Then $|\sigma^r(D)| \geq 2r - 1$.*
(ii) Let H be the graph consisting of a cycle of length n, together with n additional edges incident with all vertices of the cycle. (So $|V(H)| = 2n = |E(H)|$.) Suppose that $|E(H) - D| \geq 4r - 2$ for some $r > 0$. Then $|\sigma^r(D)| \geq 4r - 2$.

Proof. (i) Define the distance $d(e, f)$ between two edges e and f to be one less than the length of the shortest path starting with e and ending with f. For disjoint sets of edges P, Q, define the distance $d(P, Q)$ between P and Q to be the minimum, over all edges $e \in P$ and $f \in Q$ of $d(e, f)$. Among all edge sets $T \subset E(G) - D$ of size $2r - 1$, consider the one that minimizes $\sum_{e \in T} d(\{e\}, D)$. Call this set T_0. Note that T_0 exists since by our hypothesis $|E(G) - D| \geq 2r - 1$.

We will show that $T_0 \subset \sigma^r(D)$. This suffices to complete the proof, since $|T_0| = 2r - 1$. Suppose, on the contrary, that there exists an edge $e \in T_0$ such that $d(\{e\}, D) \geq r + 1$. Let P be a shortest path starting with e and ending in D, say at edge $f \in D$. Then the length of P is at least $r + 2$. Let X be the set of r edges of $P - \{f\}$ closest to f (note that $e \notin X$). By the definition of G, there is a set Y of $r-1$ edges outside of P, incident to X, and incident to neither e nor f (thus $e \notin Y$). Because of the edges of P, one can see that $X \cup Y \subset \sigma^r(D)$. By the definition of T_0, $X \cup Y \subset T_0$, since otherwise we could replace e by an edge from $X \cup Y$, contradicting the minimality of T_0. Now $|X \cup Y| = 2r-1$ and $e \in T_0 - (X \cup Y)$ leads to $|T_0| > 2r-1$, a contradiction.

(ii) We use a similar strategy to prove this part as in part (i). Fix an edge set D such that $\emptyset \neq D \subset E(H)$ and $|E(H) - D| \geq 4r - 2$. Define T_0 as in (i). Again, to get a contradiction we have to consider an edge e such that $d(\{e\}, D) \geq r + 1$. Now we may find two edge disjoint "shortest" paths from e to D. Let P^1 denote a shortest path from e to D not containing e. (This path has length at least 1 since $r > 0$.) Deleting the edges of P^1, the edge e is still in a component with some edges of D, hence there exists a shortest path P^2 connecting them, which is edge disjoint from P^1. Now repeating the argument for both paths that we did in (i) for P, we can obtain the statement (ii). $\square$

We consider the vertices of $P_n \oplus P_n$ as the elements of an $n \times n$ matrix, thus the vertex (i,j) lies in row i and column j. Call an edge *horizontal* if it is of the form $\langle (i,j), (i,j+1) \rangle$, and *vertical* if it is of the form $\langle (i,j), (i+1,j) \rangle$. The left (right) vertex of $\langle (i,j), (i,j+1) \rangle$ is (i,j) ($(i,j+1)$), and the top (bottom) vertex of $\langle (i,j), (i+1,j) \rangle$ is (i,j) ($(i+1,j)$). Define the *row* r_i (*column* c_j) to be the set of $n-1$ horizontal (vertical) edges $\langle (i,1), (i,2) \rangle, \langle (i,2), (i,3) \rangle, \ldots, \langle (i,n-1), (i,n) \rangle$ ($\langle (1,j), (2,j) \rangle, \langle (2,j), (3,j) \rangle, \ldots, \langle (n-1,j), (n,j) \rangle$). Define a *line* to be a row or a column. Let t be the smallest integer such that edges labelled $1, \ldots, t$ contain a line ℓ. Assume without loss of generality that ℓ is a row that is not r_1. Let S be the set of edges labelled by $1, \ldots, t$. Define R to be the set of rows r for which

1) there is a vertical edge from S whose bottom vertex is on r, or
2) there is a horizontal edge from S in r.

Claim 1. $|\partial(S)| \geq n + |R| - 1$.
Proof. By the definition of t, $\partial(S)$ contains one vertical edge from each column. By the definition of R, $\partial(S)$ contains one horizontal edge from each row in R except ℓ. This gives n vertical edges and $|R| - 1$ horizontal edges in $\partial(S)$. $\square$

Claim 2. $|\sigma^{n-|R|}(S)| \geq (2n-1)(n - |R|) - n$.
Proof. We begin with by associating to each column c_j (for $j > 1$), a set E_j of $2(n - |R|) - 1$ edges, such that $E_j \cap E_{j'} = \emptyset$ for $j \neq j'$.

For each vertex (i,j) for which $r_i \notin R$ and $i > 1$, we consider the two edges $e = \langle (i,j-1), (i,j) \rangle$ and $f = \langle (i-1,j), (i,j) \rangle$. In other words, these edges are the horizontal edge with right endpoint (i,j) and the vertical edge with bottom endpoint (i,j). By definition of R, neither e nor f are in S.

This way we get at least $2(n-|R|)-1$ edges outside S (we get one more edge if $r_1 \in R$). Now consider the graph G_j consisting of all $n-1$ vertical edges of the column c_j for a fixed $j=1,\dots,n$, and the $n-1$ horizontal edges $\langle(i,j-1),(i,j)\rangle$, where $i>1$. Let $D=S\cap E(G)$. Then G_j satisfies the hypothesis of Lemma 7 with $r=n-|R|$, (note that $D\neq\emptyset$ because of the row ℓ) and we conclude that $2(n-|R|)-1$ of the edges of G_j lie in $\sigma^{n-|R|}(D)$. For the column c_1 a similar argument produces $n-|R|-1$ vertical edges in $\sigma^{n-|R|}(S)$. Altogether we have produced $(n-1)(2(n-|R|)-1)+n-|R|-1=(2n-1)(n-|R|)-n$ different edges in $\sigma^{n-|R|}(S)$. □

Now if $|R|\geq n-\sqrt{n}$, then Claim 1 and Proposition 4 apply, while if $|R|\leq n-\sqrt{n}$, then Claim 2 and Proposition 6 apply. Putting these together, we get

$$B'(P_n\oplus P_n)\geq \min_{|R|}\max\left\{n+|R|-1,\frac{(2n-1)(n-|R|)-n}{n-|R|}\right\}\geq 2n-\sqrt{n}-1.$$

□

3.2. **The case of tori.** We leave the easy construction for the upper bound to the reader. The proof of the lower bound is very similar to the proof of (1). Consider an edge labelling of $C_n\oplus C_n$. Let t be the smallest integer such that the set of edges labelled by $1,2,\dots,t$ contains all but one edges of a line ℓ. Without loss of generality, ℓ is a row. Let again R be the set of rows r for which there is a vertical edge from S whose bottom vertex is on r, or there is a horizontal edge from S in r.

Claim 1^***.** $|\partial(S)|\geq 2n+2|R|-1$.

Proof. By definition of t, $\partial(S)$ contains at least two vertical edges from each of the columns and from each row in R except ℓ, and 1 from ℓ. □

Claim 2^***.** $|\sigma^{\lceil(n-|R|)/2\rceil}(S)|\geq 2n(n-|R|)$.

Proof. We begin with by associating to each column c_j a graph H_j (isomorphic to the one of Lemma 7 (ii)), and let $D:=S\cap E(H_j)$. Then $|E(H_j)-D|\geq 2(n-|R|)$, by the definition of R. This means that Lemma 7 (ii) can be applied, with $r=\lceil(n-|R|)/2\rceil$, proving the statement. (Note that we can always assume that the intersection of the row ℓ and H_j is in S, providing that D is not empty.) □

Now if $|R|\geq n-\sqrt{2n}$, then Claim 1^* and Proposition 4 apply, and if $|R|<n-\sqrt{2n}$, then Claim 2^* and Proposition 6 apply. □

4. Product of two cliques

First we demonstrate that $B'(K_n\oplus K_n)\leq 3n^3/8+19n^2/8$. To simplify the construction, we give a mapping $\eta: E(K_n\oplus K_n)\mapsto[1,\dots,n^3]$ instead of mapping the edges onto $[1,\dots,2n\binom{n}{2}]$. A further simplification is that we shall not bother with the error terms of quadratic sizes, divisibility or the exact endpoints of the subintervals of $[1,\dots,n^3]$.

Before getting into the quite painful details, let us outline the ideas behind the construction. We consider the vertices of $K_n\oplus K_n$ as cells of an $n\times n$ matrix, and the edges are among the cells of a row or column. Obviously, if we put the "smallest" numbers to the upper left part, then the "biggest" numbers have to be placed to the lower right part. So the first idea one may think is to divide the matrix into four equal sub-matrices (upper, lower, left and right), and use up the numbers for the edges in the following order. Fill first the upper left sub-matrix, then the edges between the upper and lower left sub-matrices, then the lower sub-matrix and so on.

However, one runs into great difficulties when trying to decide about the labels of edges going between the left and right side. To overcome these difficulties we use a trickier division of the matrix, and using the numbers for labelling more economically. This means to save some of the smaller numbers, and use those up only later, where the naive construction would result in too big differences. We also have to maintain a symmetry in order to keep the number of appearing cases reasonably small.

We start with explaining this symmetry first, then the labelling of edges inside a sub-matrix, finally the division and the labels among those matrices. The construction involves some optimization, that is why we had to define some strange looking numbers.

Cutting up this matrix into rectangles, the function η shall be defined on the edges inside rectangles and between two rectangles. A rectangle will be specified by its upper left and lower right corner.

The function η will be "antisymmetric" with respect to the center of the matrix, that is

$$\eta(\langle(n-i+1, n-j+1),(n-k+1, n-j+1)\rangle) = n^3 - \eta(\langle(i,k),(i,j)\rangle)$$

and

$$\eta(\langle(n-i+1, n-j+1),(n-i+1, n-k+1)\rangle) = n^3 - \eta(\langle(i,j),(i,k)\rangle).$$

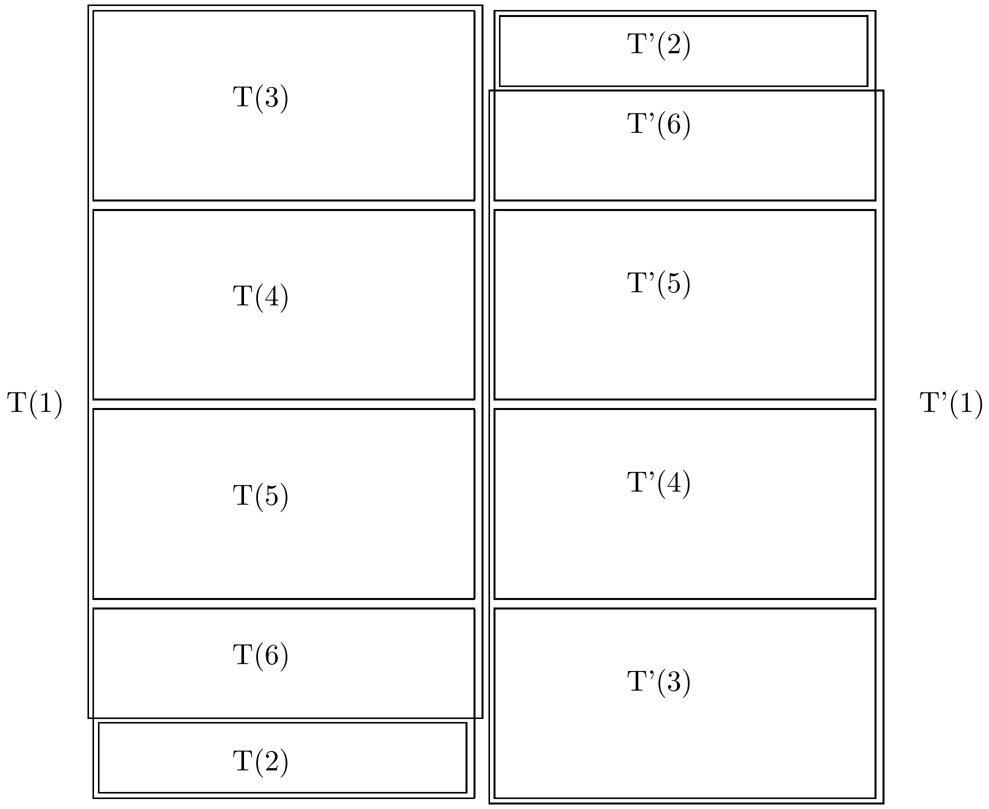
This way it suffices to define η for only half of the edges.

The first method is to assign labels from a given set of numbers I to all edges of an ℓ by k rectangle T called the *simple block*. The elements of I are used in order, starting from the smallest to fill T row by row. That is for every i, assuming that the edges (inside) of an i by k sub-rectangle T_i are labelled, then the edges connecting the vertices of the $(i+1)$st row with the vertices of T_i are labelled, finally the edges inside the $(i+1)$st line get their label. In the first case we proceed row by row, like reading a text, and order the edges connected to $(i+1, j)$ by the first coordinate of their other endpoint. The second one is done in a *lexical* way according to the second coordinates of the endpoints, i. e. the order of the labelling is $\langle(i+1,1),(i+1,2)\rangle$, $\langle(i+1,1),(i+1,3)\rangle$, $\langle(i+1,2),(i+1,3)\rangle$, and so on.

Let $a := \lceil(\sqrt{2}-1)n/4\rceil$. We shall refer to the following sub-rectangles:

- $T(1)$ with corners $(1,1)$ and $(n-a, n/2)$,
- $T(2)$ with corners $(n-a+1, 1)$ and $(n, n/2)$,
- $T(3)$ with corners $(1,1)$ and $(n/4, n/2)$,
- $T(4)$ with corners $(n/4+1, 1)$ and $(n/2, n/2)$,
- $T(5)$ with corners $(n/2+1, 1)$ and $(3n/4, n/2)$

- $T(6)$ with corners $(3n/4+1, 1)$ and $(n, n/2)$.
- Let furthermore $T'(i)$ be the centrally symmetric image of $T(i)$ for $i = 1, \ldots, 6$.



The rectangles

Figure 2

First we use the interval $[1, \ldots, n(n-a)(3n/2-a)/4]$ to make a simple block out of $T(1)$.

Next we use the interval $[n(n-a)(3n/2-a)/4+1, 23n^3/64]$ to label the edges between $T(1)$ and $T(2)$. The order is the same that we used in building the simple block, but there are no edge labels inside the rows now.

The interval $[23n^3/64, 27n^3/64]$ is used to label the edges between $T(3)$ and $T'(6)$. It is done similarly as before (going through the rows of $T'(6)$, and order the edges by the second coordinate of their other endpoint).

The most subtle part is the labelling of the edges between $T(4)$ and $T'(5)$. Now the labels are from the interval $[27n^3/64+1, 32n^3/64]$. There are $4n^3/64$ edges to be labelled, that is $n^3/64$ numbers will be saved for later use. For an edge $\langle(i,j),(i,k)\rangle$ connecting these rectangles, let us denote the smallest label occurring in (i,j) by $\eta_{i,j}$. Let $\eta(\langle(i,j),(i,k)\rangle) := 3n^3/8 + (i-1)n(n/2+i)/4 + (k-n/2) + (j-1)n/2$. To see that this part of the labelling is well-defined, three observations are needed for $n/4 < i \leq n/2,\ 1 \leq j \leq n/2 < k \leq n$:

- $\eta_{i,j+1} - \eta_{i,j} > n/4$,
- $\eta_{i+1,i} - \eta_{i,n/2} \geq n^2/4 + n/4$
- $\eta_{i+1,j} - \eta_{i,j} > n^2/4$.

To label the edges of the rectangle $T(2)$, we use up the leftover $n^3/64$ numbers of $[27n^3/64+1, 32n^3/64]$ as a simple block.

This completes the definition of the function η, as by the symmetry it was enough to define the labelling up to $n^3/2$. We need to show that the bandwidth of the labelling η is indeed less than $3n^3/8 + 19n^2/8$:

The largest differences between the labels of two edges e and f having common endpoints, up to the central symmetry, are contained in the following list:
(i) The edge e is between rectangles $T(1)$ and $T(2)$, and f is in $T(2)$. By definition, $\eta(e) \geq n(n-a)(3n/2-a)/4$ and $\eta(f) \leq n^2/2$, implying that $\eta(f) - \eta(e) < 3n^3/8$.
(ii) The edge e is inside of $T(3)$ and f is between $T(3)$ and $T'(6)$, having there common endpoint in the t-th row. Then $\eta(e) \geq (t-1)n(n/2+t-3)/4$ and $\eta(f) \leq 23n^3/64 + tn^2/4$. These bounds and an optimization in the variable t shows that
$\eta(f) - \eta(e) \leq 23n^3/64 + tn^2/8 - t^2n/4 + n^2/8 + nt - 3n/4 \leq 3n^3/8 + n^2/2$.
(iii) The edge e is in the rectangle $T(4)$ and f is between $T(4)$ and $T'(5)$, with their common endpoint in the t-th row, where $n/4 < t \leq n/2$. Then $\eta(e) \geq (t-1)n(n/2+t-3)/4$ and $\eta(f) \leq (t-1)n(n/2+t)/4 + 3n^3/8 + n/2 + (n/2-1)n/2$. It is easy to check that
$\eta(f) - \eta(e) \leq 3(t-1)n/4 + 3n^3/8 + n/2 + n^2/4 - n/2 \leq 3n^3/8 + 7n^2/8$.
(iv) The edge e is between $T(3)$ and $T'(2)$, and f is between $T'(2)$ and $T'(1)$, with their common endpoint in the t-th row, where $1 \leq t \leq a$. Then $\eta(e) \geq 23n^3/64$ and $\eta(f) \leq n^3 - n(n-a)(3n/2-a)/4 < n^3 - n \cdot 7n/8 \cdot 11n/8 \cdot (1/4) = 179n^3/256$ implying $\eta(f) - \eta(e) \leq 3n^3/8$.
(v) The edge e is between $T(3) \cup T(4)$ and $T'(1)$, and f is in $T'(1)$, with their common endpoint in the t-th row, where $a < t \leq n/2$. Then $\eta(e) \geq (t-1)n(n/2+t)/4 + 3n^3/8$ and $\eta(f) \leq n^3 - (n-t-1)n(n/2+n-t-3)/4$, implying

$$\eta(f) - \eta(e) \leq n^3/4 + tn^2/2 - t^2n/2 + 19n^2/8 \leq 3n^3/8 + 19n^2/8.$$

Proof of the lower bound.
Fix a labelling η. Consider $S := S_{n^2(n-1)/8}(\eta)$, defined as the set of edges receiving labels from $[n^2(n-1)/8]$. Let C denote the collection of columns and R the collection of rows, containing an endpoint of an edge from S. We shall give a lower bound on the cardinality of the 2-shadow of S:

An edge of S is determined by its two endpoints. The first can be chosen from the set $R \times C$, the second either from the leftover rows or columns, that is $|R| + |C| - 2$ ways. Since we have counted all edges twice and $|S| = n^2(n-1)/8$, we have

$$\frac{1}{2}|R||C|(|C| + |R| - 2) \geq \frac{n^2(n-1)}{8} = |S|.$$

This yields $(|R|+|C|)^2(|C|+|R|-2) \geq n^2(n-1)$ by the arithmetic-geometric means inequality, which implies that $|C| + |R| > n$, or $|C| + |R| \geq n+1$, because of the integrality of the left hand side.

A similar counting argument gives a lower bound on the 2-shadow $\sigma^2(S)$. From the set of all edges of $K_n \oplus K_n$, we leave out the set S and those edges having both endpoints outside of C and R.

$$|\sigma^2(S)| \geq n^2(n-1) - \frac{n^2(n-1)}{8} - (n-|C|)\binom{n-|R|}{2} - (n-|R|)\binom{n-|C|}{2}. \tag{6}$$

Note, that also by the arithmetic-geometric means inequality

$$(n-|C|)\binom{n-|R|}{2} + (n-|R|)\binom{n-|C|}{2} \leq \frac{1}{2}\Big(\frac{2n-|C|-|R|}{2}\Big)^2(2n-|C|-|R|-2).$$

Since $|C| + |R| \geq n+1$, we also have

$$\frac{1}{2}\Big(\frac{2n-|C|-|R|}{2}\Big)^2(2n-|C|-|R|-2) \leq \frac{(n-1)^2(n-3)}{8}.$$

Developing (6), and plugging in the inequalities above, one gets

$$|\sigma^2(S)| \geq \frac{3n^3}{4} - \frac{n^2}{8} - \frac{7n}{8} + \frac{3}{8},$$

that is

$$B'(K_n \oplus K_n) \geq \frac{|\sigma^2(S)|}{2} \geq \frac{3n^3}{8} - \frac{n^2}{16} - \frac{7n}{16} + \frac{3}{16}$$

by Proposition 6. □

5. The hypercube

In this section we shall prove Theorem 3. Let us start with the upper bound. First we need the following technical estimate.

Lemma 8. *Let $k \leq n$ be two integers, and fix $1 \leq i_1 < \ldots < i_k \leq n$ integers. Then*

$$(n-k)\sum_{j=1}^{k}\binom{n-i_j}{k+1-j} - (n-k-1)\sum_{j=1}^{k}\binom{n-i_j}{k+2-j} = o(n)\binom{n}{\lfloor n/2\rfloor}. \tag{7}$$

Proof. First we rewrite the left hand side of (7):

$$(n-k)\sum_{j=1}^{k}\binom{n-i_j}{k+1-j} - (n-k-1)\sum_{j=1}^{k}\binom{n-i_j}{k+2-j} =$$

$$(n-k-1)\sum_{j=1}^{k}\left\{\binom{n-i_j}{k+1-j} - \binom{n-i_j}{k+2-j}\right\} + \sum_{j=1}^{k}\binom{n-i_j}{k+1-j}.$$

We need some case analysis to handle the terms $\binom{n-i_j}{k+1-j} - \binom{n-i_j}{k+2-j}$, $j = 1, \dots k$. First assume that $i_j \geq 3\log n$. Then

$$\binom{n-i_j}{k+1-j} < 2^{n-i_j} < \frac{2^n}{n^3} < \frac{1}{n^2}\binom{n}{\lceil \frac{n}{2} \rceil}.$$

This means that in this case these terms contribute very little to the total sum.

Now we can assume that $j \leq i_j < 3\log n$. We shall use the following identity:

$$\binom{a}{b} - \binom{a}{b+1} = \frac{2b+1-a}{b+1}\binom{a}{b} = \frac{2b+1-a}{a+1}\binom{a+1}{b+1}. \tag{8}$$

If $|2k-n| < n/\log^2 n$ then using (8) we obtain

$$(n-k-1)\left\{\binom{n-i_j}{k+1-j} - \binom{n-i_j}{k+2-j}\right\} = (n-k-1)\frac{2k+2-2j-n+i_j}{k+2-j}\binom{n-i_j}{k+1-j} <$$

$$(n-k-1)\binom{n-i_j}{k+1-j}\frac{2k+3\log n - n}{k+2-3\log n} < O\Big(\frac{n}{\log^2 n}\Big)\binom{n}{\lceil \frac{n}{2} \rceil}.$$

There are at most $3\log n$ of these terms, so their contribution to the final sum is negligible. If $|2k-n| \geq n/\log^2 n$, we can use the following inequalities for $t = \lfloor n/(2\log^2 n) \rfloor$,

$$\frac{\binom{n}{\lceil n/2 \rceil + t}}{\binom{n}{\lceil n/2 \rceil}} = \frac{(n - \lceil n/2 \rceil - t + 1)\cdot \ldots \cdot (n - \lceil n/2 \rceil)}{(\lceil n/2 \rceil + 1)\cdot \ldots \cdot (\lceil n/2 \rceil + t)} < \Big(\frac{n - \lfloor n/2 \rfloor}{\lceil n/2 \rceil + t}\Big)^t =$$

$$\Big(1 - \frac{t}{\lceil n/2 \rceil + t}\Big)^t < \exp(-n/(2\log^3 n)) < \frac{1}{n^3}.$$

$\square$

Now we give a labelling of the edges of P_2^n with bandwidth $(\frac{n}{2} + o(n))\binom{n}{\lfloor n/2 \rfloor}$.

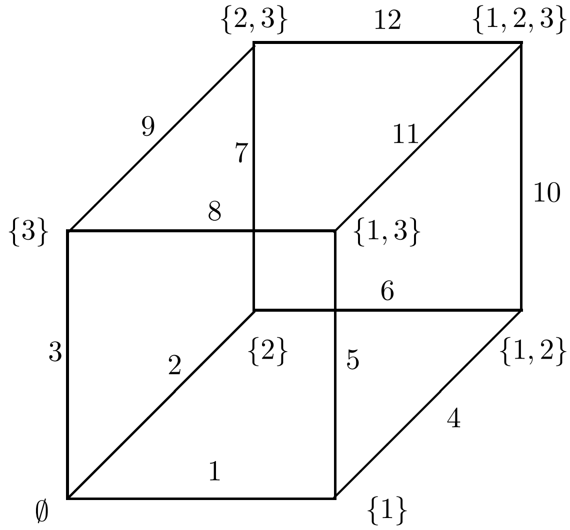
We can associate a set $A_x \subset \{1, \dots, n\}$ to any vertex x of P_2^n such that $i \in A_x$ iff the ith coordinate of x is 1. An edge can be identified with its two endpoints as $(A,\ A+e)$, where A is a subset, and $e \in \{1, \dots, n\} - A$. The labelling is done by a variant of lexicographic order, defined as follows. The first edge gets label 1, the second gets label 2 and so on. The order is:

$$(A,\ A+e) < (B,\ B+f)$$

iff one of these three conditions holds:

- $|A| < |B|$
- $|A| = |B|$ and $\min\{A \Delta B\} \in A$
- $A = B$ and $e < f$.

Remark. Indeed, this is nothing else than an appropriate breadth-first search labelling of the edges of the n dimensional cube starting from the origin. The following picture will show the details of this procedure for the three dimensional cube.



Labelling of P_2^3

Figure 3

In order to estimate the differences arising in meeting edges, we have to check the three different possibilities for the edges to meet.

(i) If the two edges are of type $(A,\ A+e)$ and $(A,\ A+f)$, then clearly there are at most $n-1$ edges between them, hence the difference of their labels is at most n.

(ii) Suppose edges of the form $(A+e, A+e+f)$ and $(A+f, A+e+f)$ meet. Without loss of generality we may assume that $e < f$. Let us estimate the number of edges of the form $(B, B+g)$, such that

$$(A+e, A+e+f) < (B, B+g) < (A+f, A+e+f).$$

The conditions above mean that $|A+e| \le |B| \le |A+f|$, from which $|A+e| = |B| = |A+f|$. For fixed A, e, f, the set B can be chosen at most $\binom{n}{|B|}$ ways, and when B is fixed, g can be chosen at most $n-|B|$ ways. Altogether the difference of the labels of $(A+e, A+e+f)$ and $(A+f, A+e+f)$ is at most

$$(n-|B|)\binom{n}{|B|} \le \left\lceil \frac{n}{2} \right\rceil \binom{n}{\lceil \frac{n}{2} \rceil}.$$

(iii) Finally we consider edges of the form $(A,\, A+e)$ and $(A+e,\, A+e+f)$. Again, we need to estimate the number of edges between these two edges in the given order.

Let $(A,\, A+e) < (B,\, B+h) < (A+e,\, A+e+f)$, where $A = \{i_1, \ldots, i_k\}$. Observe that the vertex $A+e$ has $k+1$ neighbors of size k, and the first one among them in our ordering is $\{A+e\} - \min\{A+e\}$. Hence the difference of the labels of the vertices $A+e$ and A is the largest (i.e. the number of edges $(B, B+h)$ between $(A, A+e)$ and $(A+e, A+e+f)$ satisfying $|B| = k$ is maximized) when e is maximal possible, therefore we may assume $i_k < e$. If $|B| = k$, then $\min\{A\Delta B\} \in A$ and the number of such edges is

$$(n-k)\sum_{j=1}^{k}\binom{n-i_j}{k+1-j}.$$

If $|B| = k+1$, then $\min\{\{A+e\}\Delta B\} \in B$ and the number of such edges is

$$(n-k-1)\binom{n}{k+1} - (n-k-1)\Big\{\sum_{j=1}^{k}\binom{n-i_j}{k+2-j} + \binom{n-e}{e}\Big\}.$$

Consequently, the number of edges between $(A,\, A+e)$ and $(A,\, A+f)$ is

$$(n-k-1)\binom{n}{k+1} + (n-k)\sum_{j=1}^{k}\binom{n-i_j}{k+1-j} - (n-k-1)\Big\{\sum_{j=1}^{k}\binom{n-i_j}{k+2-j} + \binom{n-e}{e}\Big\}.$$

Now, by Lemma 8,

$$(n-k)\sum_{j=1}^{k}\binom{n-i_j}{k+1-j} - (n-k-1)\sum_{j=1}^{k}\binom{n-i_j}{k+2-j} = o(n)\binom{n}{\lceil n/2\rceil}$$

and by the inequality (which could be easily checked)

$$(n-k-1)\binom{n}{k+1} \le (n/2)\binom{n}{\lceil n/2\rceil}.$$

the upper bound is proved.

The proof of the other direction is a refinement of the proof of Calamoneri, Massini and Vrťo [3]. As we shall follow their proof and notation also, first we give the sketch of their ideas, too.

They picked an arbitrary edge set S of size $n2^{n-2}$ from $E(P_2^n)$, and showed that either $|\partial(S)|$ or $|\partial(E(P_2^n) - S)|$ is at least of size $\frac{n}{4}\binom{n}{\lceil n/2\rceil}$, then applied Proposition 4. (Note that $n2^{n-2} = |E(P_2^n)|/2$.) We make a more subtle case analysis: if say $|\partial(S)|$ is greater than $\frac{n}{2}\binom{n}{\lceil n/2\rceil}$, then we are done by Proposition 4, while in the other case we take $\partial^2(S)$, $\partial^3(S)$ and so on, and use Proposition 6.

So let us fix an edge labelling of P_2^n. Let S denote the set of edges labelled by $\{1, 2, \ldots, n2^{n-2}\}$, and color the edges in S by red, and rest of the edges by white. For a vertex $x \in V(P_2^n)$, let $E(x)$ denote the set of edges incident to x. Call a vertex x red if every edge in $E(x)$ is red, white if every edge in $E(x)$ is white, and the rest is mixed. Let R, W and M denote the set of red, white and mixed vertices, respectively. Certainly,

$$|R| + |W| + |M| = 2^n. \tag{9}$$

For $x \in M$, let $r(x)$ denote the number of red edges in $E(x)$, that is $1 \le r(x) \le n-1$. Furthermore, by (double) counting the red edges we have that

$$|R| \cdot n + \sum_{x \in M} r(x) = n \cdot 2^{n-1} = |W| \cdot n + \sum_{x \in M} (n - r(x)). \tag{10}$$

From the definition of mixed vertices we can conclude the following two inequalities:

$$\frac{1}{2} \sum_{x \in M} (n - r(x)) \le |\partial(S)| \quad \text{and} \quad \frac{1}{2} \sum_{x \in M} r(x) \le |\partial(E(P_2^n) - S)|. \tag{11}$$

Combining these two inequalities we obtain

$$\frac{|M| \cdot n}{4} \le \frac{|\partial(S)| + |\partial(E(P_2^n) - S)|}{2} \le \max\{|\partial(S)|, |\partial(E(P_2^n) - S)|\}.$$

If $2\binom{n}{\lceil n/2 \rceil} \le |M|$, then by Proposition 4 we prove the required lower bound. From now on we therefore assume

$$|M| < 2\binom{n}{\lceil \frac{n}{2} \rceil}. \tag{12}$$

Either

$$\sum_{x \in M} r(x) \le |M| n/2 \quad \text{or} \quad \sum_{x \in M} (n - r(x)) \le |M| n/2;$$

let us assume that the first inequality holds, since otherwise we could switch the role of the red and white vertices. Combining this with (10) and (12) we obtain that

$$n \cdot 2^{n-1} \le |R| \cdot n + \frac{1}{2}|M| \cdot n < |R| \cdot n + n \cdot \binom{n}{\lceil \frac{n}{2} \rceil},$$

implying the lower bound

$$2^{n-1} - \binom{n}{\lceil \frac{n}{2} \rceil} < |R| < 2^{n-1}. \tag{13}$$

Note that the upper bound on $|R|$ in (13) follows from (10).

Combining (12) and (13) we obtain an upper bound for $|R \cup M|$. The lower bound given below follows from $|W| < 2^{n-1}$.

$$2^{n-1} < |R \cup M| \le 2^{n-1} + 2 \cdot \binom{n}{\lceil \frac{n}{2} \rceil}. \tag{14}$$

We need the following easy observation on the ℓ-shadows:

$$\cup_{x\in\sigma^\ell(R\cup M)} E(x) \subset \sigma^{\ell+1}(S). \tag{15}$$

To estimate $|\cup_{x\in\sigma^\ell(R\cup M)} E(x)|$ we need a classical result of Harper [7] (see also a proof of it by Katona [12]).

Lemma 9. *Let $A \subset V(P_2^n)$, $0 \le y < n$ and r be an integer such that* [1]

$$|A| = \binom{n}{n} + \binom{n}{n-1} + \ldots + \binom{n}{r+1} + \binom{y}{r}.$$

Then

$$|\partial(A)| \ge \binom{n}{r} + \binom{y}{r-1} - \binom{y}{r} \ge \binom{n}{r-1}.$$

First, for some $0 \le y_0 < n$ and an integer r_0 we have

$$|R\cup M| = \binom{n}{n} + \binom{n}{n-1} + \ldots + \binom{n}{r_0+1} + \binom{y_0}{r_0}.$$

By (14) we have that $\lceil n/2\rceil - 4 \le r_0 \le n/2$. We shall apply Lemma 9 first to the set $R \cup M$ then repeatedly to

$$R\cup M\cup\sigma(R\cup M),\ R\cup M\cup\sigma^2(R\cup M), \ldots,\ R\cup M\cup\sigma^{\ell-1}(R\cup M)$$

for $\ell = \lceil n^{1/3}\rceil - 4$. To do so, for $1 \le t \le \ell - 1$ write

$$|R\cup M\cup\sigma^t(R\cup M)| = \binom{n}{n} + \binom{n}{n-1} + \ldots + \binom{n}{r_t+1} + \binom{y_t}{r_t}, \tag{16}$$

where $y_t \le n$ and r_t is an integer.

By Lemma 9,

$$|\sigma\big(R\cup M\cup\sigma^t(R\cup M)\big)| \ge \binom{n}{r_t-1}. \tag{17}$$

Note that the sequence $\{r_t\}$ is monotone decreasing, and as $r \le n/2$ it means that the sequence $\binom{n}{r_t-1}$ is monotone decreasing also. If $r_{\ell-1} \ge \lceil n/2\rceil - \ell - 3$ then by (17)

$$|\sigma^\ell(R\cup M)| \ge \sum_{t=0}^{\ell-1}\binom{n}{r_t-1} \ge \ell\cdot\binom{n}{\lceil\frac{n}{2}\rceil-\ell-4}.$$

If $r_\ell < \lceil n/2\rceil - \ell - 3$, then taking the difference of (14) and (16) (these are disjoint sets) we have

$$|\sigma^\ell(R\cup M)| \ge \binom{n}{\lceil n/2\rceil-1} + \ldots + \binom{n}{r_\ell+1} - 2\binom{n}{\lceil n/2\rceil} > \binom{n}{\lceil n/2\rceil-4} + \ldots + \binom{n}{r_\ell+1}$$

[1] Note that for a real y, $\binom{y}{r}$ is defined as $y\cdot(y-1)\cdot\ldots\cdot(y-r+1)/r!$.

and we can conclude that

$$|\sigma^\ell(R \cup M)| \geq \ell \cdot \binom{n}{\lceil \frac{n}{2} \rceil - \ell - 3}. \tag{18}$$

(Note that we assume that n is large.)

That is by Proposition 6, (18) and (15) we have

$$B'(P_2^n) \geq \frac{n \cdot |\sigma^\ell(R \cup M)|}{2(\ell+1)} \geq \frac{n}{2}\Big(1 - \frac{1}{\ell+1}\Big)\binom{n}{\lceil \frac{n}{2} \rceil - \ell - 4}.$$

We estimate the rightmost expression of the inequality above with $\ell = \lceil n^{1/3} \rceil - 4 = t - 4$:

$$\frac{\binom{n}{\lceil n/2 \rceil - t}}{\binom{n}{\lceil n/2 \rceil}} = \frac{(n - \lceil n/2 \rceil - t + 1) \cdot \ldots \cdot (n - \lceil n/2 \rceil)}{(\lceil n/2 \rceil + 1) \cdot \ldots \cdot (\lceil n/2 \rceil + t)} > \Big(\frac{n - \lceil n/2 - t + 1 \rceil}{\lceil n/2 + 1 \rceil}\Big)^t \geq$$

$$\Big(1 - \frac{t}{n/2 + 1}\Big)^t \approx 1 - 2n^{-1/3}.$$

This proves the lower bound of the theorem. □

6. Remarks

We believe that in Theorem 1 the upper bounds are the real values of the edge-bandwidths. Alas, it is hard even to conjecture the exact value of $B'(K_n \oplus K_n)$; we have no good candidate for this.

On the other hand, it is reasonable to think that the upper bound, and the labelling given in the proof of Theorem 3 is optimal. Still, to show this will require more refined methods.

Finally, let T_ℓ be the graph whose vertices are the triples of non-negative integers summing to ℓ, with an edge connecting two triples if they agree in one coordinate and differ by 1 in the other two coordinates. Hochberg, McDiarmid and Saks [9] showed in a beautiful paper that $B(T_\ell) = \ell + 1$. It is natural to raise the following question.

Problem 10. *What is the value of $B'(T_\ell)$?*

Acknowledgment. We would like to thank the useful advices of the anonymous referees.

References

[1] J. Balogh, J. A. Csirik, Index assignment for two-channel quantization, *IEEE Trans. Inform. Theory* **11** (2004) 2737–2751.

[2] S. Bezrukov, N. Grünwald and K. Weber, On edge numberings of the n-cube graph, *Discrete Appl. Math.* **46** (1993) 99–116.

[3] T. Calamoneri, A. Massini and I. Vrťo, New results on edge-bandwidth, *Theoret. Comput. Sci.* **307** (2003) 503–513.

[4] P. Z. Chinn, J. Chvátalová, A. K. Dewdney and N. E. Gibbs, The bandwidth problem for graphs and matrices—a survey. *Journal of Graph Theory* **6** (1982), 223–254.
[5] F. R. K. Chung, Labelings of graphs, in: L. W. Beineke, R.J. Wilson (Eds.) *Selected topics in graph theory* Vol. **3**, Academic Press, San Diego, CA, (1988) 151–168.
[6] J. Chvátalová, Optimal labelling of a product of two paths. *Discrete Math.* **11** (1975) 249–253.
[7] L. H. Harper, Optimal numberings and isoperimetric problems on graphs, *J. Combinatorial Theory* **1** (1966) 385–393.
[8] L. H. Harper, Global Methods for Combinatorial Isoperimetric Problems, 232 pp., Cambridge University Press.
[9] R. Hochberg, C. McDiarmid and M. Saks, On the bandwidth of triangulated triangles, *Discrete Math.* **138** (1995) 261–265.
[10] F. K. Hwang and J. C. Lagarias, Minimum range sequences of all k-subsets of a set, *Discrete Math.* **19** (1977) 257–264.
[11] T. Jiang, D. Mubayi, A. Shastri and D. B. West, Edge-bandwidth of graphs, *SIAM J. Discrete Math.* **12** (1999) 307–316.
[12] G. O. H. Katona, The Hamming-sphere has minimum boundary, *Studia Sci. Math. Hungar.* **10** (1975) 131–140.
[13] Y. L. Lai and K. Williams, A survey of solved problems and applications on bandwidth, edge-sum, and profile of graphs, *Journal of Graph Theory,* **31** (1999) 75–94.
[14] Q. Li, M. Q. Tao and Y. Q. Shen, The bandwidth of the discrete tori $C_m \times C_n$, *J. China Univ. Sci. Tech.* **11** (1981) 1–16.
[15] C. H. Papadimitriou, The NP-completeness of the bandwidth minimization problem, *Computing* **16** (1976) 263–270.

Department of Mathematical Sciences, The Ohio State University, Columbus, OH 43210, USA
E-mail address: jobal@math.ohio-state.edu

Department of Mathematics, Statistics, and Computer Science, University of Illinois, Chicago, IL 60607, USA
E-mail address: mubayi@math.uic.edu

University of Szeged, Department of Computer Science, Árpád tér 2., Szeged, H-6720
E-mail address: pluhar@inf.u-szeged.hu