On the existence of finite isomorphically complete systems of automata*

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Abstract

In the theory of compositions of finite automata it is a central problem to study systems from which every automaton can be built under a given composition and representation. Such systems are called complete with respect to the considered composition and representation. From practical and theoretical points of view, those compositions and representations have great importance for which there are finite complete systems. Under the isomorphic embedding as representation, a graph theoretical necessary condition of the existence of finite complete systems is presented in [3]. In this paper, we give a sufficient condition for a composition to admit a finite complete system under the isomorphic representation, which reduces the question to a graph coloring problem.

1 Introduction

The notion of complete system was introduced by V. M. Glushkov in [4] where he introduced the general product and characterized the complete systems for this product under the isomorphic embedding as representation.

One can consider compositions as networks of automata. In this case, the underlying graphs are the complete graphs, and each vertex contains an automaton. If the network receives an external input sign, then simultaneously, each component automaton receives an input sign which may depend on the external input sign and all of the actual states of the ancestor automata of the considered one. On the basis of this network approach, we can define different compositions by giving the set of the available underlying graphs. Having this general definition of composition it is natural to look for conditions on the underlying graphs under which there are finite isomorphically complete systems with respect to the corresponding product. This question is studied in [3] where it is proved that if there exists a finite complete system for some composition under isomorphic embedding as representation, then for every integer k there is a graph among the underlying graphs determining the composition such that it has a connected subgraph for which the indegree of each vertex is at least k. This result immediately implies that most of the known compositions have no finite

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complete system under the isomorphic embedding. There are only two exceptions, namely, the general product [4] and the cube-product [5]. Paper [3] does not contain any sufficient condition for the underlying graphs which would imply the existence of a finite complete system under isomorphic embedding. Here we present a sufficient condition such that compositions determined by graphs satisfying this condition are equivalent to the general product with respect to the isomorphic completeness. Unfortunately, there is a gap between the necessary and sufficient conditions, and therefore it remains an open problem to characterize those underlying graphs which determine compositions admitting finite complete systems. Our result raises the question whether a composition admitting a finite complete system is equivalent to the general product with respect to the isomorphic completeness.

The paper is organized as follows. The necessary notions and notations are introduced in Section 2. The next part, Section 3 contains our main result based on the observation that there exists a finite isomorphically complete system for a given composition if the corresponding underlying graphs have a special coloring. Section 4 presents some conditions for the existence of the required special graph coloring. On the basis of these conditions we can conclude some earlier results as easy consequences.

2 Preliminaries

By an *automaton*, we mean a couple $\mathbf{A} = (X, A)$, where A and X are finite nonempty sets, the set of *states* and the set of *input symbols*, respectively, and for all $x \in X$, x is realized as a unary operation on A which is denoted by $x^{\mathbf{A}}$. Since we consider an automaton as a unoid (universal algebra with unary operations), the notions such as isomorphism, homomorphism, and subautomata can be introduced in a natural way.

Let $D = (V, E)$ be a directed graph consisting of a nonempty finite set of vertices $V = \{1, \ldots, n\}$ and a set $E \subseteq V \times V$ of edges, and let us consider an arbitrary nonempty set $\mathcal{D}$ of such finite directed graphs. Furthermore, let $\mathbf{A}_j = (X_j, A_j)$, $j = 1, \ldots, n$, be a system of automata, X a nonempty finite set and φ a mapping of $A_1 \times \cdots \times A_n \times X \to X_1 \times \cdots \times X_n$. An automaton $\mathbf{A} = (X, A)$ is called a *$\mathcal{D}$-product* of $\mathbf{A}_j$, $j = 1, \ldots, n$, if the conditions below are satisfied:

(1) $A = \prod_{j=1}^{n} A_j$.

(2) There exists a graph $D = (\{1, \ldots, n\}, E)$ in $\mathcal{D}$ such that the mapping φ can be given in the following form:

$$\varphi(a_1, \ldots, a_n, x) = (\varphi_1(a_1, \ldots, a_n, x), \ldots, \varphi_n(a_1, \ldots, a_n, x))$$

for all $(a_1, \ldots, a_n) \in A$, $x \in X$, moreover φ_j is independent of any a_i with $(i, j) \notin E$ for every j, $j = 1, \ldots, n$.

(3) For all $(a_1, \ldots, a_n) \in A$ and $x \in X$,

$$(a_1, \ldots, a_n)x^{\mathbf{A}} = (a_1 x_1^{\mathbf{A}_1}, \ldots, a_n x_n^{\mathbf{A}_n})$$

where $x_j = \varphi_j(a_1, \ldots, a_n, x)$, $j = 1, \ldots, n$.

For the product introduced above we use the notation

$$\prod_{j=1}^{n} \mathbf{A}_j(X, \varphi, D).$$

Now, let Γ be a system of automata. Γ is called an *isomorphically complete system with respect to the $\mathcal{D}$-product* if every automaton can be embedded isomorphically into a $\mathcal{D}$-product of automata from Γ.

It is worth noting that we obtain the notion of the general product if the set $\mathcal{D}$ of the underlying graphs is the set of the complete graphs of the form $D_n = (\{(1,\ldots,n\},\{1,\ldots,n\} \times \{1,\ldots,n\})$, $n = 1, 2, \ldots$.

3 Existence of finite isomorphically complete systems

The following statement shows that if a finite graph can be colored in an appropriate way, then the composition having this graph as its underlying graph is strong enough.

Lemma 1 *Let $D = (\{1,\ldots,n\}, E)$ be a directed graph and also let $\mathbf{A} = \prod_{j=1}^{m} \mathbf{A}_j(X, \varphi)$ be a general product of automata where $1 \le m < n$. Moreover, let $\chi : \{1,\ldots,n\} \to \{1,\ldots,m\}$ be such a coloring of the vertices of D that for every vertex $r \in \{1,\ldots,n\}$ $\chi(S_r) = \{1,\ldots,m\}$, where S_r denotes the set of the ancestors of r in D. Then $\mathbf{A}$ can be embedded isomorphically into a $\mathcal{D}$-product of copies of $\mathbf{A}_j$, $j = 1,\ldots,m$, with the underlying graph D.*

Proof. Let us denote the elements of $\prod_{j=1}^{m} \mathbf{A}_j(X, \varphi)$ by $(a_{t1},\ldots,a_{tm})$, $t = 1,\ldots s$, where $s = |\prod_{j=1}^{m} A_j|$. Then the elements $(a_{t1},\ldots,a_{tm})$, $t = 1,\ldots,s$, are pairwise different. Without loss of generality, it can be assumed that the copies of each automaton $\mathbf{A}_j$ occurring in the considered general product are distinguished. In this case, the sets $\{a_{t1},\ldots,a_{tm}\}$, $t = 1,\ldots,s$, are pairwise different, and thus there exist functions φ'_j, $j = 1,\ldots,m$, such that for all $j \in \{1,\ldots,m\}$, $x \in X$ and $1 \le t \le s$,

$$\varphi_j(a_{t1},\ldots,a_{tm}, x) = \varphi'_j(\{a_{t1},\ldots,a_{tm}\}, x)$$

is valid. Let us define the mapping μ of the set $\prod_{j=1}^{m} A_j$ into the set $\prod_{i=1}^{n} A_{\chi(i)}$ as follows.

$$\mu : (a_{t1},\ldots,a_{tm}) \to (a_{t,\chi(1)},\ldots,a_{t,\chi(n)})$$

for all t, $t = 1,\ldots,s$ and let $C = \{(a_{t,\chi(1)},\ldots,a_{t,\chi(n)}) : t = 1,\ldots,s\}$. Since $\chi(S_r) = \{1,\ldots,m\}$ for all $r \in \{1,\ldots,n\}$, μ is a one-to-one mapping of $\prod_{j=1}^{m} A_j$ onto C. Now, let us define the $\mathcal{D}$-product $\mathbf{B} = \prod_{i=1}^{n} \mathbf{A}_{\chi(i)}(X, \bar{\varphi}, D)$ in the following way. For all $i \in \{1,\ldots,n\}$, $x \in X$ and $(a_1,\ldots,a_n) \in \prod_{i=1}^{n} A_{\chi(i)}$, let

$$\bar{\varphi}_i(a_1,\ldots,a_n,x) = \begin{cases} \varphi'_{\chi(i)}(\{a_{t1},\ldots,a_{tm}\},x) & \text{if } (a_1,\ldots,a_n) = (a_{t,\chi(1)},\ldots a_{t,\chi(n)}) \\ & \text{for some } 1 \le t \le s, \\ \text{an arbitrarily fixed } x \in X & \text{otherwise.} \end{cases}$$

Since the sets $\{a_{t1}, \ldots, a_{tm}\}$, $t = 1, \ldots, s$, are pairwise different, the functions $\bar{\varphi}_i$, $i = 1, \ldots, n$, are well-defined. Moreover, since $\chi(S_r) = \{1, \ldots, m\}$, for all $r \in \{1, \ldots, n\}$, the defined product is a $\mathcal{D}$-product.

Finally, we prove that C determines a subautomaton in $\prod_{i=1}^{n} \mathbf{A}_{\chi(i)}(X, \bar{\varphi}, D)$ which is an isomorphic image of $\mathbf{A}$ under the mapping μ. For this purpose, let $(a_{t1}, \ldots, a_{t_m}) \in \prod_{j=1}^{m} A_j$ and $x \in X$ be arbitrary elements. Then it is enough to prove that

$$(a_{t1}, \ldots, a_{tm})x^{\mathbf{A}}\mu = (a_{t\chi(1)}, \ldots, a_{t,\chi(n)})x^{\mathbf{B}}.$$

Let us suppose that $(a_{t1}, \ldots, a_{tm})x^{\mathbf{A}} = (a_{q1}, \ldots, a_{qm})$ for some $q \in \{1, \ldots, s\}$. Then the required equality is valid if $a_{t,\chi(i)} x_i^{\mathbf{A}_{\chi(i)}} = a_{q,\chi(i)}$ for all $i \in \{1, \ldots, n\}$, where $x_i = \bar{\varphi}_i(a_{t,\chi(1)}, \ldots, a_{t,\chi(n)}, x)$. Let $i \in \{1, \ldots, n\}$ be arbitrary. By the definition of $\bar{\varphi}_i$,

$$x_i = \bar{\varphi}_i(a_{t,\chi(1)}, \ldots, a_{t,\chi(n)}, x) = \varphi'_{\chi(i)}(\{a_{t1}, \ldots, a_{tm}\}, x),$$

moreover $\varphi'_{\chi(i)}(\{a_{t1}, \ldots, a_{tm}\}, x) = \varphi_{\chi(i)}(a_{t1}, \ldots, a_{tm}, x)$. Let $\chi(i) = j$. Then the left side of the required equality is $a_{tj} x_i^{\mathbf{A}_j}$, where $x_i = \varphi_j(a_{t1}, \ldots, a_{tm}, x)$. On the other hand, by the definition of the general product, $a_{tj} x_j^{\mathbf{A}_j} = a_{qj}$ where $x_j = \varphi_j(a_{t1}, \ldots, a_{tm}, x)$. Let us observe that $x_i = x_j$, and consequently $a_{t,\chi(i)} x_i^{\mathbf{A}_{\chi(i)}} = a_{q,\chi(i)}$ for all $i = 1, \ldots, n$. This completes the proof of Lemma 1. □

The next statement presents a sufficient condition for a nonempty set $\mathcal{D}$ of directed graphs for admitting the existence of a finite isomorphically complete system.

Theorem 1 *Let $\mathcal{D}$ be a nonempty set of finite directed graphs of the form given at the beginning of the previous section. Then, there exists a finite isomorphically complete system with respect to the $\mathcal{D}$-product, if for every positive integer m, $\mathcal{D}$ contains a graph D which has a connected subgraph $D' = (V', E')$ having a coloring $\chi : V' \to \{1, \ldots, m\}$ of the vertices of D' such that, for every vertex $r \in V'$, $\chi(S_r) = \{1, \ldots, m\}$ where S_r denotes the set of the ancestors of r in D'.*

Proof. Let $\mathbf{B}_2 = (\{w, x, y, z\}, \{0, 1\})$ be the automaton which is defined by $0w^{\mathbf{B}_2} = 1z^{\mathbf{B}_2} = 0$ and $1y^{\mathbf{B}_2} = 0x^{\mathbf{B}_2} = 1$. To prove our statement, we shall show that an arbitrary automaton can be embedded isomorphically into a suitable $\mathcal{D}$ product of suitable copies of $\mathbf{B}_2$. For this purpose, let $\mathbf{A} = (X, A)$ be an arbitrary automaton. By the theorem of Glushkov [4], $\mathbf{A}$ can be embedded isomorphically into a general product $\prod_{j=1}^{m} \mathbf{B}_2(X, \varphi)$. Let us distinguish the copies of $\mathbf{B}_2$ as $\mathbf{A}_1, \ldots, \mathbf{A}_m$. Furthermore, let $\mathbf{A}_{m+1}$ denote a further copy of $\mathbf{B}_2$. By our assumption, there is a graph $D = (V, E) \in \mathcal{D}$ containing a subgraph $D' = (V', E')$ which has a coloring $\chi : V' \to \{1, \ldots, m\}$ such that for every vertex $r \in V'$, $\chi(S_r) = \{1, \ldots, m\}$. Without loss of generality, it can be assumed that $V' = \{1, \ldots, n\}$ for some positive integer n, furthermore, let $|V| = s$. Consider the function $\rho : V \to \{1, \ldots, m+1\}$ given in the following way. For every $j \in \{1, \ldots, s\}$ let

$$\rho(j) = \begin{cases} \chi(i) & \text{if } 1 \leq j \leq n, \\ m+1 & \text{otherwise.} \end{cases}$$

Now, let us define the $\mathcal{D}$-product $\prod_{i=1}^{s} \mathbf{A}_{\rho(i)}(X, \bar{\varphi}, D)$ as follows. For the first n components, let the product be given in the same way as in the proof of Lemma 1, and for the last $s-n$ components, let us choose the state 0, and let the value of $\bar{\varphi}_j$, $n < j \leq s$, be w. Then it can be seen that $\prod_{j=1}^{m} \mathbf{B}_2(X, \varphi)$, and thus also $\mathbf{A}$, can be embedded isomorphically into the $\mathcal{D}$-product $\prod_{i=1}^{s} \mathbf{A}_{\rho(i)}(X, \bar{\varphi}, D)$ which ends the proof of Theorem 1. □

By the theorem of Glushkov [4], a system Γ of automata is isomorphically complete with respect to the general product if and only if there exists an $\mathbf{A} \in \Gamma$ such that $\mathbf{B}_2$ can be embedded isomorphically into a general product of $\mathbf{A}$ with a single factor. Thus, by the proof of Theorem 1, if Γ is isomorphically complete with respect to the general product, then Γ is isomorphically complete with respect to the $\mathcal{D}$-product given in Theorem 1. The converse assertion is obvious since the $\mathcal{D}$-product is a special case of the general product. Consequently, we obtain the following assertion.

Corollary 1 *If a $\mathcal{D}$-product satisfies the condition of Theorem 1, then it is equivalent to the general product regarding isomorphic completeness.*

4 On colorings

First we need some definitions of Combinatorics. An $H = (V(H), E(H))$ is a *hypergraph* if $V(H) = X$ (the *ground set* of H) and $E(H) \subseteq 2^X$ (the *edge set* of H). A hypergraph H is *k-colorable* if there exists a $\chi : V(H) \to \{1, ..., k\}$ such that $| \cup_{x \in E} \chi(x)| > 1$ for all $E \in E(H)$, i.e. every edge of H receives at least two colors. Furthermore, $\chi(H)$ stands for the smallest k for which H is k-colorable. Notice that to decide if $\chi(H) = 2$ is an NP-complete problem.

In order to utilize the power of Lemma 1 and Theorem 1, one must prove their colorability assumption for certain classes of graphs. Since we are dealing with directed graphs, this assumption readily translates to hypergraph coloring, which is more convenient to study. Indeed, for a finite directed graph $D = (\{1, ..., n\}, E)$, we can define the hypergraph $H = (V(H), E(H))$, where the ground set $V(H) = \{1, ..., n\}$ and the edge set $E(H) = \{S_r : 1 \leq r \leq n\}$. Obviously, what we need is to decide if there exists a coloring $\chi : V(H) \to \{1, ..., m\}$ such that $|\chi(E)| = m$, for all $E \in E(H)$, where m is a given positive integer. Upon existence of such a coloring, we call the hypergraph H *m-colorful.* Here we have to realize that we cannot expect good characterization for this colorability problem because every hypergraph can be obtained in this way, and the problem is NP-complete even for $m = 2$ since H is 2-colorful if and only if it is 2-colorable.

Still, one can show the existence of the required coloring in some cases even though the involved hypergraphs have no nice structure. The following assertion is a more or less straightforward generalization of the corresponding 2-colorability results and methods.

Lemma 2 *Let H be a hypergraph such that $|E| \geq k$ for all $E \in E(H)$, $|E(H)| = n$ and $k > 2m \log n$. Then H is m-colorful.*

Proof. Let us color the elements of $V(H)$ randomly and independently of each other, giving the color i to an element with probability $1/m$ for $1 \leq i \leq m$. Furthermore, A_E denotes the event in this product measure that $E \in E(H)$ does not receive all colors and $Pr(A_E)$ denotes the probability of A_E as usual. Because $|E| \geq k$, $Pr(A_E) \leq m(1-1/m)^k \leq me^{-k/m}$. Since $k/m > 2\log n \geq \log n + \log m$, we have

$$\sum_{E \in E(H)} Pr(A_E) \leq nme^{-k/m} < 1.$$

That is $V(H)$ can be colored such that all $E \in E(H)$ receives all colors $1, ..., m$. $\square$

Lemma 2 and Theorem 1 immediately give one of the results in [3], which states that a general product of automata can be embedded isomorphically into a $\mathcal{D}$-product if $\mathcal{D}$ contains large, "almost" complete graphs. Precisely, there exist a universal constant c and a sequence of graphs $D_{n_l} \in \mathcal{D}$ such that $|V(D_{n_l})| = n_l$ and the number of the ancestors of every vertex of D_{n_l} is at least $n_l - c$.

Lemma 2 shows that this result can be extended in the following way. Let $f(n)$ be a function which tends to ∞ if $n \to \infty$. If every vertex has at least $f(n)\log n$ ancestors, then among the corresponding underlying graphs there exists an m-colorful for every positive integer m.

We note that Lemma 2 cannot be significantly sharpened in general, since the hypergraph H, where $V(H) = \{1, ..., 2k-1\}$ and $E(H)$ consists of all k-element subsets of $V(H)$ is not even 2-colorable. For certain important graph classes, like cubes (see [5]) Lemma 2 extends. Not surprisingly the celebrated Lovász' Local Lemma is the vehicle of the extension. We need a definition to spell out the Local Lemma. If $A_1, ..., A_n$ are events of a probability space, then a *dependence graph* $G = (V, E)$ of these events is a graph having the following properties: $V = \{1, ..., n\}$, and each event A_i is mutually independent of the events $\{A_j : (i,j) \notin E\}$. For details see [1] and [2].

Lemma 3 (Lovász Local Lemma) [2] *Let $A_1, ..., A_n$ be events of a probability space, and G be a dependence graph of these events. If $Pr(A_i) \leq p$, $deg_G(A_i) \leq d$ and $epd < 1$ for all $1 \leq i \leq n$, then $Pr(\cap_{i=1}^{n} \overline{A}_i) > 0$.*

Let H be a hypergraph. For all $E \in E(H)$, let $|\{F : F \in E(H), F \cap E \neq \emptyset\}| = d(E)$ and $d = \max\{d(E) : E \in E(H)\}$. Then the following statement is valid.

Lemma 4 *Let H be a hypergraph and k a positive integer such that for all $E \in E(H)$, $|E| \geq k$. If $k > m(1 + \log d + \log m)$, then H is m-colorful.*

Proof. One just checks that using the same random coloring as in the proof of Lemma 2, $Pr(A_E) \leq p = me^{-k/m}$, and A_E is independent of the sigma-algebra generated by the events A_F, where $F \cap E = \emptyset$. Thus Lemma 3 applies, and gives the result, since

$$epd \leq eme^{-k/m}d < emde^{-(1+\log d+\log m)} = 1,$$

and thus, $epd < 1$ holds. $\square$

As an application of Lemma 4, let us consider the hypercubes as underlying graphs (see [5]). Let $n > 0$ be an arbitrary integer. Then, $|S_{\mathbf{r}}| = n$ for any vertex $\mathbf{r} \in \{0,1\}^n$ of the n-dimensional hypercube. Moreover, let us observe that $S_{\mathbf{r}} \cap S_{\mathbf{r}'} \neq \emptyset$ if and only if $\mathbf{r}$ and $\mathbf{r}'$ are different exactly in two components. Consequently, $d = d(S_{\mathbf{r}}) = n(n-1)/2$, for all $\mathbf{r} \in \{0,1\}^n$. Now, if m is a fixed positive integer, then we can choose an n such that

$$n > m(1 + \log \frac{n(n-1)}{2} + \log m)$$

is valid. Therefore, by Lemma 4, the n-dimensional hypercube is m-colorful, and consequently, by Corollary 1, the cube-product is equivalent to the general product regarding the isomorphically complete systems which is the main result in [5].

Let us observe that on the basis of Lemma 1 and Lemma 4, we can obtain a stronger statement. Namely, if $\prod_{j=1}^{m} \mathbf{A}_j(X, \varphi)$ is a general product of automata, then by Lemma 4, there exists a positive integer n such that the n-dimensional hypercube is m-colorful. Then, by Lemma 1, the considered general product can be embedded isomorphically into a cube-product of the copies of the automata $\mathbf{A}_j$, $j = 1, \ldots, m$. This implies that the cube-product is equivalent to the general product which is the main result in [6] where this statement is proved by a constructive coloring of the vertices of the coresponding hypercube.

It looks very plausible that somehow the graph colorings are responsible for *all* of the complexity of the general product of automata. One may formalize this assertion in the following conjecture.

Conjecture 1 *If a product determined by the set $\mathcal{D}$ of underlying graphs is equivalent to the general product regarding isomorphic completeness, then $\mathcal{D}$ contains a sequence of graphs $\{D_m\}_{m=1}^{\infty}$ having subgraphs $\{D'_m\}_{m=1}^{\infty}$ such that D'_m is m-colorful.*

Finally, we note that the notion of product and completeness can be extended to tree automata (see e.g. [7]), and one can extend these results to tree automata, too.

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