The Interval Number of Dense Graphs*

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Abstract

The interval number of a graph G, denoted by $i(G)$, is the least natural number t such that G is the intersection graph of sets, each of which is the union of at most t closed intervals. Most known bounds on $i(G)$ are grossly excessive when G has more than half of the possible edges. A plausible remedy is to develop bounds on $i(G)$ that are monotone decreasing in G. Here we bound $i(G)$ in terms of $e(\bar{G})$, the number of edges in the complement of G. We prove that $i(G) \leq \lceil \frac{1}{2}\sqrt{e(\bar{G})}\,\rceil + O(n/\log n)$.

1 Introduction

In this paper we study special intersection representations of graphs. We assign a set to each vertex of G so that v is adjacent to w if and only if the assigned set intersect. A *t-interval representation* is such an assignment in which each set consists of the union at most t closed intervals. The *interval number* of G, denoted by $i(G)$, is the least integer t such that a t-representation of G exists. A t-interval representation is *displayed* if the set assigned to each vertex contains an open interval (called a *displayed segment*) that is disjoint from the other assigned sets. Various aspects of the interval number have been studied. Sharp bounds are known in terms of the number of vertices ($v(G)$), number of edges ($e(G)$), the maximum degree ($\Delta(G)$), and the maximum density of a graph. The maximum density is defined as $\rho(G) = \max_{H \subset G} e(H)/v(H)$.

Theorem 1 *[4]* $i(G) \leq \lceil(\Delta(G)+1)/2\rceil$.

Theorem 2 *[3]* $i(G) \leq \lceil(v(G)+1)/4\rceil$.

Theorem 3 *[2] Every graph G has a displayed t-interval representation with $t \leq \lceil\frac{1}{2}\sqrt{e(G)}\rceil + 1$.*

Theorem 4 *[7, 2] Every graph G has a displayed t-interval representation with $t \leq \lceil\rho(G)\rceil + 1$. Moreover, G has an orientation in which the maximum outdegree is no more than $\lceil\rho(G)\rceil$.*

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These bounds are attained for the complete bipartite graph $K_{2m,2m}$ with m a positive integer. (See [4] and [9] for $i(K_{m,m}) = \lceil (m+1)/2 \rceil$.)

The message of all these bounds is that when a graph is sparse, its interval number is small. However, the interval number of a dense graph is not necessarily large; in particular $i(K_n) = 1$. A bound for very special graphs was proved in [1]: $i(\bar{T}) \leq 3$ when T is a tree. Several bounds in terms of the complement should be investigated.

Conjecture 1 *[2]* $i(G) \leq \frac{1}{2}\lceil \sqrt{e(\bar{G})} \rceil + 1$.

This bound would be sharp, since it is attained for the complete bipartite graph $K_{2m,2m}$ and for the graph obtained by adding an edge from each vertex of $K_{2m,2m}$ to each vertex of a clique of order t. These are the only extremal graphs known so far. Note that Conjecture 1 and Theorem 3 would essentially imply Theorem 2.

Here we take a step toward settling Conjecture 1. As a byproduct, we also obtain upper bounds in terms of $\Delta(\bar{G})$.

Theorem 5 $i(G) \leq \lceil \frac{1}{2}\sqrt{e(\bar{G})} \rceil + (3/2 + o(1))n/\log n$.

Theorem 6 $i(G) \leq \lceil (\Delta(\bar{G}) + 1)/2 \rceil + 2^{n/(\Delta(\bar{G})+1)}$.

Theorem 7 $i(G) \leq \Delta(\bar{G}) + \frac{1}{2}\chi(\bar{G}) + 1$, *where* χ *denotes the chromatic number.*

Corollary 1 $i(G) \leq \frac{3}{2}(\Delta(\bar{G}) + 1)$.

Perhaps every bounds in terms of density has an analogue in terms of the complement.

Conjecture 2 $i(G) \leq \lceil \frac{1}{2}(\Delta(\bar{G}) + 1) \rceil$ *and* $i(G) \leq \lceil \rho(\bar{G}) \rceil + 1$.

2 Proofs of the Results

It is costumary in this subject to speak of assigning individual intervals to a vertex instead of assigning the set that is their union. In our representations, we first establish disjoint displayed intervals in a unused portion of the line, assigning one to each vertex. We refer to these intervals as the *basic intervals*. Often we will represent an edge uv by assigning a small interval for u in the interior of the basic interval for v. This adds an interval for u to the representation, but does not alter the set assigned to v.

Observation 0: Given basic intervals for $V(G)$ and an orientation of a subgraph of G, one can represent the oriented edges using at most $d^+(x)$ additional intervals for each vertex x.

Proof: For each oriented edge xy, we assign a small interval for x within the basic interval for y. (This technique first appeared in [7], although it was observed earlier by Trotter.) □

We next consider optimal proper colorings of $\bar{G}$. With every proper coloring of $\bar{G}$ with $\chi(\bar{G})$ colors, we associate a $\chi(\bar{G})$-dimensional vector in which the components are the sizes of the color classes. A *lexicographically minimal* coloring of $\bar{G}$ is a proper coloring with $\chi(\bar{G})$ colors for which the vector of sizes is lexicographically minimal. Within a lexicographically minimal coloring, let c_1, c_2, c_3 count the color classes with one element, two element, or at least three elements, respectively; these are the *parameters* of the coloring.

Lemma 1 *For a lexicographically minimal coloring of $\bar{G}$ with parameters c_1, c_2, c_3,*

$$e(\bar{G}) \geq \binom{c_1}{2} + 2\binom{c_2}{2} + 3\binom{c_3}{2} + c_1c_2 + c_1c_3 + 2c_2c_3.$$

Proof: In a lexicographically minimal coloring, every vertex has neighbor in each color class that is at least as big as its own; otherwise, its color can be changed to that of the larger class to improve the size vector. The expression claimed is the result of summing these requirements. □

We shall partition the edge set of G and represent the edges in these classes separately, almost independently. First we consider the part of G that is induced by the color classes of size at most two in a lexicographically minimal coloring of $\bar{G}$.

Lemma 2 *Given basic intervals, one can represent the subgraph H of G induced by the vertices in color classes of size at most two in a lexicographically minimal coloring of $\bar{G}$ using at most $p = \lceil \frac{c_2(c_1+c_2)}{c_1+2c_2} \rceil$ additional intervals per vertex.*

Proof: First we show that $\rho(H_1) \leq p$ for all $H_1 \subset H$. Let $d_1 \leq c_1$ be the number of one-element classes in H that lie in H_1. Let $d_2 \leq c_2$ be the number of two-element classes in H that contribute at least one vertex to H_1, with l of them contributing exactly one vertex and $d_2 - l$ contributing both. Lexicographic minimality of the coloring implies that a vertex from one of the incomplete classes in H_1 cannot be connected by two edges in G to one of the complete classes. Similarly, the number of edges induced by the union of the complete 2-element classes is at most $2\binom{d_2-l}{2} + (d_2 - l) = (d_2 - l)^2$. Now we can bound $e(H_1)$, much like in the proof of Lemma 1. We compute

$$e(H_1) \leq ld_1 + d_1(d_2 - l) + l(d_2 - l) + (d_2 - l)^2 = d_1d_2 + (d_2 - l)d_2 =$$

$$d_2(d_1 + d_2 - l) \leq d_2(d_1 + 2d_2 - l)\frac{d_1 + d_2}{d_1 + 2d_2} = (d_1 + 2(d_2 - l) + l)\frac{d_2(d_1 + d_2)}{d_1 + 2d_2} \leq v(H_1)p.$$

That is $\lceil \rho(H) \rceil \leq p$, so Theorem 4 implies Lemma 2. □

Scheinerman [8] introduced a generalization of the displayed representation method. Consider a clique-partition of the vertices of a graph G. View cliques as vertex sets. For each clique Q in the partition, establish $2^{|Q|}$ disjoint intervals in an unused portion of the line. For each subset of Q, assign the corresponding interval to each member of that subset.

This *generalized displayed representation* increases the number of intervals used x by 2^{k-1}, where k is the size of the clique containing x. Given such a system, we can represent the edges between the cliques P and Q cheaply; for a vertex $x \in P$ we just use a small interval within the system of identical intervals for the neighborhood of x in Q. This way of representing all edges of a (P,Q) pair uses one interval for each vertex of P and none for Q; we call this method a *Scheinerman representation.*

Lemma 3 *If k is the maximum size of a color class in a lexicographically minimal coloring of $\bar{G}$ with parameters c_1, c_2, c_3, then*

$$i(G) \leq \left(\frac{c_2(c_1+c_2)}{c_1+2c_2}(c_1+c_2) + (c_1+c_2)c_3 + \binom{c_3}{2} \right) \frac{1}{c_1+c_2+c_3} + 2^{k-1} + 3.$$

Proof: The color classes in the coloring of $\bar{G}$ yield a clique partition of G. We establish generalized displayed representations for the cliques with at least three vertices. The main problem is how to choose the ordered pairs of cliques for representation of edges between them in the Scheinerman representation.

If we process an ordered pair (P,Q), then we need not process the pair (Q,P). Also, nothing needs to be done for pairs of cliques that both have size at most 2; using Lemma 2, these edges are being represented in H using p intervals per vertex. We aim to orient the $(c_1+c_2)c_3 + \binom{c_3}{2}$ leftover pairs so that each clique appears first not too often. We start the count at zero for a clique of size at least three and p for a clique of size at most two.

In effect, consider the graph J that is the join of K_{c_3} with c_1+c_2 vertices that are independent of each other but each incident to p loops. We orient J to minimize the maximum outdegree, so that each vertex has essentially the average outdegree. (Since $p < c_2$, the average exceeds p, so one can see that $\lceil \rho(J) \rceil \leq e(J)/v(J)$. Then the second part of Theorem 4 applies.)

This orientation yields the claimed formula. We wrote a $+3$ to ignore the ceiling on p and ρ, and account for the basic intervals for small cliques in case there are no large cliques. The 2^{k-1} intervals in the generalized displayed representation of large cliques can include the basic intervals for those vertices. □

Lemma 4 *If k is the maximum size of a clique in a lexicographically minimal coloring of $\bar{G}$, then $i(G) \leq \frac{1}{2}\sqrt{e(\bar{G})} + 2^{k-1} + 4$.*

Proof: We compare the key quantity in Lemma 3 with the number of edges in $\bar{G}$. Let

$$f = \left(\frac{c_2(c_1+c_2)}{c_1+2c_2}(c_1+c_2) + (c_1+c_2)c_3 + \binom{c_3}{2} \right) \frac{1}{c_1+c_2+c_3}.$$

Observe that

$$\frac{2c_2(c_1+c_2)^2}{c_1+2c_2} = (c_1+c_2)^2 - \frac{c_1(c_1+c_2)^2}{c_1+2c_2}.$$

Thus

$$2f = \left(\frac{2c_2(c_1+c_2)^2}{c_1+2c_2} + 2(c_1+c_2)c_3 + c_3^2 - c_3\right)\frac{1}{c_1+c_2+c_3} \leq$$

$$\left((c_1+c_2)^2 + 2c_1c_3 + 2c_2c_3 + c_3^2 - \frac{c_1(c_1+c_2)^2}{c_1+2c_2}\right)\frac{1}{c_1+c_2+c_3} =$$

$$c_1 + c_2 + c_3 - \frac{c_1(c_1+c_2)^2}{(c_1+c_2+c_3)(c_1+2c_2)}.$$

To obtain an useful bound on $(2f)^2$, we need to show that

$$\frac{2c_1(c_1+c_2)^2}{c_1+2c_2} - \frac{c_1^2(c_1+c_2)^4}{(c_1+c_2+c_3)^2(c_1+2c_2)^2} \leq c_1(c_1+c_2).$$

To this end it suffices to consider $c_3 = 0$, get rid of fractions, and compare terms. As a result,

$$(2f)^2 \leq (c_1+c_2+c_3)^2 - c_1(c_1+c_2).$$

Rearranging the bound on $e(\bar{G})$ from Lemma 1 yields

$$e(\bar{G}) \geq (c_1+c_2+c_3)^2 - c_1(c_1+c_2) + \frac{1}{2}(c_1-c_3)^2 - \frac{1}{2}(c_1+2c_2+c_3).$$

It is easy to check by Lemma 1 that $c_1 + 2c_2 + c_3 \leq 3\sqrt{e(\bar{G})}$. From here we obtain

$$(2f)^2 \leq e(\bar{G}) + \frac{3}{2}\sqrt{e(\bar{G})} \leq e(\bar{G}) + 2\sqrt{e(\bar{G})} + 1,$$

and thus $2f \leq 1 + \sqrt{e(\bar{G})}$. The result now follows from Lemma 3. □

When k is large, this bound is not very good; what can we do with the large cliques? We cut them into small pieces to obtain the main theorem.

Proof of Theorem 5: If the size of a color class in the lexicographically minimal coloring of $\bar{G}$ exceeds s, a value to be chosen later, then we cut it into sets of sizes between $s/2$ and s. This yields at most $2n/s$ new sets; we consider one piece from each color class to be an old set. Let G' be the graph induced by the small classes and the old pieces. Representing it as in Lemma 3 uses at most $\frac{1}{2}\sqrt{e(\bar{G})} + 2^{s-1} + 3$ intervals per vertex.

There are at most $2n/s$ new pieces; we apply the Scheinerman representation. We establish generalized displayed representations for the cliques that are the new pieces. We represent the edges between old and new pieces using at most $2n/s$ additional intervals for each vertex in an old piece, and we represent the edges between new pieces using at most n/s additional intervals for each vertex in a new piece. This yields

$$i(G) \le \frac{1}{2}\sqrt{e(\bar{G})} + 2^{s-1} + 2 + 2n/s,$$

which implies Theorem 5 by setting $s = (\log n - \log\log n - \log\log\log n)/\log 2$. $\square$

Theorem 8 *[5] If G is an n-vertex graph with $\Delta(G)$ less than an integer l, then $V(G)$ can be partitioned into l independent sets whose size differ by at most 1.*

Proof of Theorem 6: Theorem 6 follows from Theorem 8. Applying the method of Theorem 5 to a coloring of $\bar{G}$, described in Theorem 8, yields Theorem 6. $\square$

Proof of Theorem 7: An optimal coloring of $\bar{G}$ yields a partition of $V(G)$ into $\chi(\bar{G})$ cliques in G. We represent each clique using one interval per vertex.

In order to represent the edges from a vertex x to a clique Q, we can place a small interval for x into the clique for Q and cut a hole in the intervals for each vertex of Q that is not adjacent to x. When we do this with many vertices, the interval for a vertex $z \in Q$ breaks into at most $\Delta(\bar{G})$ parts.

As usual, we distribute the ordered pairs of cliques uniformly, so that we insert an interval for vertex x into at most $\frac{1}{2}\chi(\bar{G})$ other cliques. This gives the claimed bound. $\square$

Proof of Corollary 1: We need to show that $i(G) \le \frac{3}{2}(\Delta(\bar{G})+1)$. This immediately follows from Theorem 7, since $\chi(\bar{G}) \le \Delta(\bar{G}) + 1$. $\square$

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