The positive minimum degree game on sparse graphs

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Abstract

In this note we investigate a special form of degree games defined by D. Hefetz, M. Krivelevich, M. Stojaković and T. Szabó [4]. Usually the board of a graph game is the edge set of K_n, the complete graph on n vertices. Maker and Breaker alternately claim an edge, and Maker wins if his edges form a subgraph with prescribed properties; here a certain minimum degree. In the special form the board is no longer the whole edge set of K_n, Maker first selects as few edges of K_n as possible in order to win, and our goal is to compute the necessary size of that board. Solving a question of [4], we show, using the discharging method, that the sharp bound is around $10n/7$ for the positive minimum degree game.

1 Introduction

We briefly recall the necessary definitions about positional games, for a deeper introduction see J. Beck [3] or specifically D. Hefetz, M. Krivelevich, M. Stojaković and T. Szabó [4]. The game $\mathcal{H} = (V(\mathcal{H}), E(\mathcal{H}))$ is played by two players, called Maker and Breaker, on the hypergraph $\mathcal{H} = (V(\mathcal{H}), E(\mathcal{H}))$. The players take turns in claiming one previously unclaimed vertex of $V(\mathcal{H})$; here we assume Breaker starts the game and the game ends if all vertices are taken. Maker wins by taking every element of some $A \in E(\mathcal{H})$, while Breaker wins if he can take at least one vertex of every edge in $E(\mathcal{H})$. Clearly, exactly one of the players wins this game.

We consider *graph games*; in those $V(\mathcal{H}) \subset E(K_n)$, and $E(\mathcal{H}) \subset 2^{E(K_n)}$. The set $E(\mathcal{H})$ is a *global property* if for all $A \in E(\mathcal{H})$ the subgraph spanned by A has $\Omega(n)$ vertices. A

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well-known example for this is the *Shannon's switching game,* in which $E(\mathcal{H})$ consists of the edge sets of the spanning trees in $V(\mathcal{H})$, [5]. Here Maker wins if and only if $V(\mathcal{H})$ hosts two edge-disjoint spanning trees. That is $2n-2$ appropriately selected edges results in a Maker's win, while Breaker wins if $V(\mathcal{H})$ has fewer than $2n-2$ edges. In general, for a property $\mathcal{P} = E(\mathcal{H})$, we look for $\hat{m}(\mathcal{P})$, the smallest size of $V(\mathcal{H})$ that is sufficient to guarantee Maker's win.

In this note we investigate a global property, achieving in a graph a given minimum degree. A possible way to define degree games is to say $\mathcal{D}_k := \mathcal{D}_k(n)$ consist of the subgraphs of K_n having minimum degree k. For other degree games see [1, 2, 3, 7]. D. Hefetz, M. Krivelevich, M. Stojaković and T. Szabó [4] showed that $11n/8 \leq \hat{m}(\mathcal{D}_1) \leq 10n/7 + 4$, and $\hat{m}(\mathcal{D}_1) \leq 10n/7$ if $n = 7\ell$.

Our main result is the exact determination of $\hat{m}(\mathcal{D}_1)$.

Theorem 1 *For $n \geq 4$*

$$\left\lceil \frac{10}{7}n \right\rceil = \hat{m}(\mathcal{D}_1), \text{ for } n \not\equiv 2 \pmod 7,$$

and

$$\left\lceil \frac{10}{7}n \right\rceil + 1 = \hat{m}(\mathcal{D}_1), \text{ for } n \equiv 2 \pmod 7.$$

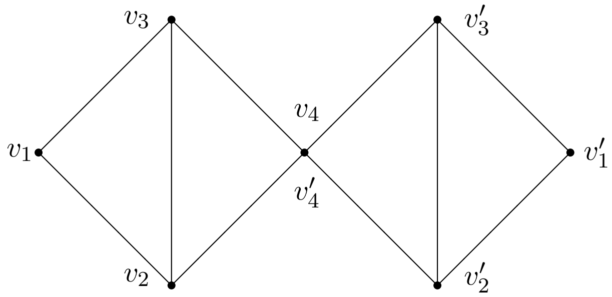
It means that there exists graphs with $\lceil \frac{10}{7}n \rceil$ (or $\lceil \frac{10}{7}n \rceil + 1$, if $n = 7\ell + 2$) edges on n vertices on which Maker wins, and on graphs with fewer edges always Breaker wins. Similar, but much weaker results might be spelled out for $\hat{m}(\mathcal{D}_2)$ and $\hat{m}(\mathcal{D}_k)$.

Proposition 2 *For all $k \in \mathbb{N}$, $\hat{m}(\mathcal{D}_2) \leq \frac{20n}{7}$ for $n = 14k$, and $\hat{m}(\mathcal{D}_k) \leq \frac{10}{7}kn$ for $n = 7^k$.*

L. Székely [7] and J. Beck [2] studied degree games on the edges of $K_{n,n}$, but their methods also apply to K_n. Briefly Maker wins if $k < n/2 - \sqrt{n \log n}$, and Breaker wins if $k > n/2 - \sqrt{n}/12$. To put it differently, if k is about $n/2 - \sqrt{n \log n}$, then $\hat{\mathcal{D}}_k = (k + f(k))n$, where $f(k) = O(\sqrt{k \log k})$. It would be interesting to know if the function f behaves similarly for smaller values, too. A quick calculation shows that adding copies of $K_{\ell,\ell}$ and using a *weight function* strategy on each $K_{\ell,\ell}$ gives a better bound on $\hat{m}(\mathcal{D}_k)$ than $10kn/7$, provided $k > 32$.

2 Proofs

Proof of Theorem 1. Observe that for $n \leq 3$ Breaker always wins, so we may assume that $n \geq 4$. First we construct graphs on which Maker can win, establishing the upper bounds. It is easy to check that Maker wins on $K_4, K_{3,3}$, and D_7, where D_7 is a "double diamond", that is we take two copies of $K_4 \setminus v_1v_4$, and identify the two vertices called "v_4." (Note that this was observed already in [4].)



The graph D_7. Here $v_4 = v_4'$.

For $n = 7k$, Maker wins on k vertex disjoint copies of D_7. For $n = 7k+4$, Maker wins on k vertex disjoint copies of D_7 and one copy of K_4. Similarly, for $n = 7k+6$, Maker wins on k vertex disjoint copies of D_7 and one copy of $K_{3,3}$. For the rest of the cases, the following observation is sufficient: Assume that Maker wins on G. Form G' by adding a vertex to G, with edges to arbitrarily chosen two other vertices. Then by playing the winning strategy on the edges of G and a simple pairing on the two newly selected edges, Maker wins on G' as well. For $n = 7k+1$ Maker wins on $(k \cdot D_7)'$, for $n = 7k+2$ on $((k \cdot D_7)')'$, for $n = 7k+3$ on $(k-1) \cdot D_7 + K_4 + K_{3,3}$, and for $n = 7k+5$ on $(k \cdot D_7 + K_4)'$.

Now we prove the lower bounds. Let G be such a graph on n vertices with as few edges as possible, that Maker, as a second player, can achieve degree at least one at every vertex. We first observe some properties of G.

A path $xz_1 \dots z_m y$ is a *(k, ℓ)-(x, y)-path*, if $d(x) = k$, $d(z_1) = \dots = d(z_m) = 3$ and $d(y) = \ell$. Note that m can be 0, that is the path consisting of the edge xy.

Lemma 3 *The graph G has the following properties.*
(i) For every $x \in V(G)$ we have $d(x) \geq 2$.
(ii) There is no $(2,2)$-(x,y)-path in G.
(iii) For any $k \geq 3$, there are no vertices $x, y_1, \dots, y_{k-1}$ with a $(k,2)$-(x,y_i)-path for every i.
(iv) There is a vertex $x \in V(G)$ with $d(x) = 2$.
(v) In a component of G if there is a vertex $x \in V(G)$ with $d(x) = 2$ then either there is a vertex $y \in V(G)$ with $d(y) \geq 4$ or the component consists of at least 7 vertices.

Proof of Lemma 3. If (i) was false then Breaker would trivially win instantly.

To prove (ii), assume for a contradiction that G contains a $(2,2)$-(x,y)-path $xz_1 \dots z_m y$, where m is chosen to be minimum possible. The minimality of m implies that this path is an induced path. Now Breaker easily wins, claiming edges $xz_1, z_1z_2, \dots, z_m y$. In order to avoid instant loss, Maker has to claim the last unclaimed edge at $x, z_1, \dots, z_m$, and finally Breaker could claim the last unclaimed edge at y.

For (iii), assume that for some $k \geq 3$ such a path system exists. We might choose one, with minimum number of vertices. Then using (ii) and the minimality, any edge spanned by the vertex set of this path system is also an edge of some of those paths. Now as in the proof of part (ii), Breaker can claim all path edges, starting from the vertices y_i, and then could claim the last edge at x.

Part (iv) follows from that if G was a counterexample for Theorem 1, then its density was smaller than 3/2.

For (v) assume that G is a counterexample. Then by (ii), each component of G contains at most one vertex of degree 2. Each such component should have odd number of vertices. There is one possible graph with 5 vertices, and it is easy to see that Breaker wins on it. □

We now apply the *discharging method*, see e. g. [6]:

In the charging phase a vertex $v \in V(G)$ is assigned a weight or *charge* $w(v) := d(v)$.

In the discharging phase the vertices send some of their charges to other, not necessarily neighboring vertices. The rules of the discharging are as follows:

1. A vertex of degree 2 sends no charge.
2. Only vertices of degree 2 receive any charge.
3. A 3-degree vertex x sends charge 1/7 to a 2-degree vertex y if there is a $(3, 2)$-(x, y)-path.
4. If for a $k > 3$ there is a $(k, 2)$-(x, y)-path, then x sends a charge of 4/7 to y.

In the beginning the sum of the charges is the sum of the degrees. The sum of the charges does not change during the discharging phase, so the following claim completes the proof for the case when $n \neq 7\ell + 2$.

Claim 1 *After the discharging phase every vertex has charge at least* 20/7.

Proof of Claim 1. Observe that charges are staying within components. If every vertex of a component is of degree at least 3, then the charges do not change, and the claim is trivially true. If a component has a degree-2 vertex but has no vertex of degree at least 4, then by Lemma 3 (v) it consists of at least 7 vertices. By Lemma 3 (ii) such a component contains exactly one vertex of degree 2. Each vertex of degree 3 sends a charge of 1/7 to the 2-degree vertex, which will have charge at least $2 + 6 \cdot 1/7 = 20/7$. The 3-degree vertices will have charge $3 - 1/7 = 20/7$.

Now consider a component, which contains vertices both of degree 2 and at least 4. By Lemma 3 (iii), a vertex x with $d(x) = 3$ sends to at most one 2-vertex a charge of 1/7, so it will have charge at least $3 - 1/7 = 20/7$. For any $k > 3$, by Lemma 3 (iii), a vertex x with $d(x) = k$ will have charge at least

$$k - 4(k - 2)/7 = 3k/7 + 8/7 \geq 20/7. \tag{1}$$

Now assume that $d(x) = 2$. For some $k > 3$ there is a $(2, k)$-(x, y) path for some k-vertex y. So x will receive from y a charge of 4/7. If $xy \notin E(G)$ then x has two neighbors of degree 3, and receives charges of 1/7 from both. So the charge of x will be at least

$2 + 4/7 + 1/7 + 1/7 = 20/7$. Otherwise, let z be the other neighbor of x. Observe, that $d(z) \geq 3$, because of Lemma 3 (ii), and z should send a charge of at least $1/7$ to x. Observe also that the vertex z must have a neighbor w that differs both from x and y, and, by Lemma 3 (ii) the degree of w is at least 3. But then w sends a charge at least $1/7$ to x, which is sufficient to achieve charge of at least $20/7$ at x. □

The proof of Theorem 1 is completed when $n \not\equiv 2 \pmod 7$.

Assume that $n = 7k + 2$ for some positive integer k. Then the sum of the charges in G is at least $(7k + 2) \cdot 20/7 = 20k + 40/7$, yielding that $e(G) \geq 10k + 3$. If we were able to find extra $3/7$ charges, then this would be sufficient to imply $e(G) \geq 10k + 4$, and then the proof of Theorem 1 would have been completed.

Trivially, each component contains at least four vertices. If there is a component, with each of its vertices having degree at least 3, then each will have extra charge $1/7$ and the proof is completed. Now we can assume that each component contains a degree-2 vertex. If there is a vertex of degree at least 5, then (1) implies that this vertex will have at least extra $3k/7 + 8/8 \geq 20/7 + 3/7$ charge. So now we can assume that the maximum degree is at most 4. If there is a component which contains only vertices of degree 2 and 3, then Lemma 3 (iii) implies that it contains only one degree-2 vertex, and by parity reasons $2s$ degree-3 vertices for some s. One can check (we omit the details) that Breaker wins if $s \leq 4$. On the other hand, if $s \geq 5$ then we have extra charge at least $2s/7 - 6/7 \geq 4/7$. So now we can assume that each component contains vertices of both degree 2 and 4.

Since $n \equiv 2 \pmod 7$, G must have a component C, such that $|C| \not\equiv 0 \pmod 7$.

Assume now that in such a component C there are $\ell \geq 1$ vertices of degree 4. Then there should be exactly 2ℓ vertices of degree 2. It cannot be fewer, otherwise either a degree-4 vertex retains charge $4/7$ or at least two degree-2 vertices overcharged by $2/7$, or a degree-2 vertex is overcharged by $6/7$ and we are done. It cannot be more, as now every degree-2 vertex receives charge from a degree-4 vertex, and a degree-4 vertex can send charges to at most two degree-2 vertices.

The proof of Claim 1 implies that each degree-2 vertex receives charges from one degree-4 vertex and at least two degree-3 vertices. So the number of degree-3 vertices is at least 4ℓ. Because $|C| \not\equiv 0 \pmod 7$, it is more than 4ℓ, but it must be less than $4\ell + 4$, otherwise there would be $4/7$ extra charge, So the number of degree-3 vertices is $4\ell + 2$.

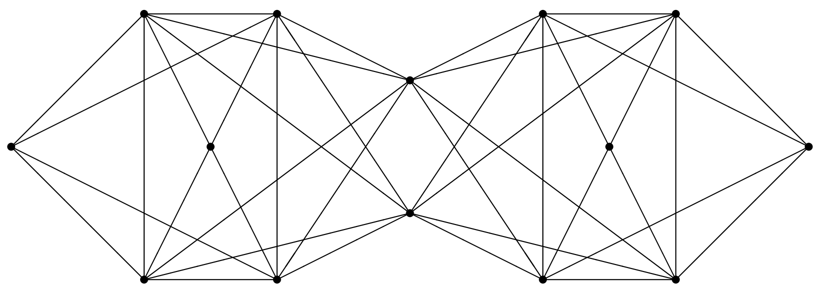
Denote $\{x_1, \ldots, x_\ell\}$ the set of degree-4 vertices in C. Observe that $C - \{x_1, \ldots, x_\ell\}$ has at least 2ℓ components, and 2ℓ of them contain a degree-2 vertex and at least two degree-3 vertices. Assume that $2\ell - i$ components contain exactly two degree-3 vertices among those, where $i \leq 2$. There are at least $4\ell - i$ edges between these 2ℓ components and the vertex set $\{x_1, \ldots, x_\ell\}$, and because C is connected, the number of additional edge endpoints of vertices in $\{x_1, \ldots, x_\ell\}$ is at least ℓ, yielding $4\ell - i + \ell \leq 4\ell$, i.e. $\ell \leq 2$. Note that when there is a cut-edge, then Breaker in his first step can claim it, and can easily win in the component in which Maker does not occupy an edge. One can check that when $\ell = 2$ then there is always a cut-edge. When $\ell = 1$, then C has nine vertices, with degree sequence $2, 2, 3, 3, 3, 3, 3, 3, 4$, and the degree-4 vertex is a cut-vertex. It can be checked that Breaker has a winning strategy, we omit the details. □

Sketch of the proof of Proposition 2.

We use the graph D_7 that was shown to be a Maker's win in the positive minimum degree game in [4]. From two copies of D_7 one can make a graph D_{14} such that $v(D_{14}) = 14$, $e(D_{14}) = 40$ and Maker wins the game $(E(D_{14}), \mathcal{D}_2)$. The construction is made in two steps.

First we take two copies of D_7, and glue them together in three vertices. The degree two and degree four vertices are associated to the same vertices in the other copy, one to each other. The resulting graph H has 11 vertices and 20 edges. Note that playing Maker's winning strategy for the positive minimum degree game separately on the edges of the D_7's, Maker gets degrees at least two at the "glued" vertices, and at least one at the others.

In the second step we glue together two copies of H, this time at the degree three vertices, taking care not creating parallel edges, this we call D_{14}. (Simply the vertices of a diamond should be glued to vertices that are not in a diamond in the other copy.) Again, playing separately on the edges of the two copies of H, Maker gets at least two degrees at all vertices of D_{14}.



The graph D_{14}.

Let $D_7^2 = D_7 \Box D_7$ be the Cartesian product of D_7 with itself, and $D_7^k := D_7 \Box D_7^{k-1}$ be kth power of D_7. To play the $(E(D_7^k), \mathcal{D}_k)$-game, let Maker play the winning strategy for a $(E(D_7), \mathcal{D}_1)$-game in the same projection in that Breaker has just played. This clearly gives a winning strategy for the $(E(D_7^k), \mathcal{D}_k)$-game, and $e(D_7)/v(D_7) = 10(7)^{k-1}k/7^k = 10k/7$. $\Box$

Remarks. The discharging method also gives lower bound on $\hat{m}(\mathcal{D}_2)$, alas, it is not matching with $\hat{m}(\mathcal{D}_2) \leq 20n/7$, the upper bound of Proposition 2. In fact, we think the upper bound on $\hat{m}(\mathcal{D}_2)$ is not tight, but we cannot improve on it.

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