GREEDY COLORINGS OF UNIFORM HYPERGRAPHS

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Abstract. We give a very short proof of an Erdős conjecture that the number of edges in a non-2-colorable n-uniform hypergraph is at least $f(n)2^n$, where $f(n)$ goes to infinity. Originally it was solved by József Beck in 1977, showing that $f(n)$ at least $c \log n$. With an ingenious recoloring idea he later proved that $f(n) \geq cn^{1/3+o(1)}$. Here we prove a weaker bound on $f(n)$, namely $f(n) \geq cn^{1/4}$. Instead of recoloring a random coloring, we take the ground set in random order and use a greedy algorithm to color. The same technique works for getting bounds on k-colorability. It is also possible to combine this idea with the Lovász Local Lemma, reproving some known results for sparse hypergraphs (e.g., the n-uniform, n-regular hypergraphs are 2-colorable if $n \geq 8$).

1. Introduction

We use the notation of [2] and partly those of [8]. A hypergraph (V, E) is k-colorable if V can be colored by using at most k colors such that no edge $A \in E$ is monochromatic. Let $m_k(n)$ denote the minimum possible number of edges of an n-uniform hypergraph that is non-k-colorable. We suppress the lower index for $k = 2$, that is $m(n) = m_2(n)$. We list some of the significant results concerning the values of $m_k(n)$ and other colorability issues as follows.

Erdős proved lower and upper bounds on $m(n)$, namely $2^{n-1} \leq m(n) \leq cn^2 2^n$ in [5] and in [6], respectively. While the upper bound is still the best known, the lower bound on $m(n)$ was improved in a sequence of papers. Note that all subsequent works start with his idea, that is coloring the vertices randomly and independently of each other.

First Schmidt showed $(1 - 2/n)2^n < m(n)$ (see [11]), then Beck came up with the idea of *recoloring* of a random coloring, and he proved the bounds $c \log n 2^n < m(n)$ and $n^{1/3+o(1)} 2^n < m(n)$ [3, 4], respectively. The proof of the latter bound was simplified by Spencer in [12].

Twenty years later Radhakrishnan and Srinivasan modified the recoloring idea of Beck, and showed $0.7\sqrt{n/\ln n}2^n < m(n)$ [10]. In the same paper it was shown that a hypergraph is 2-colorable if every edge meets at most $0.17\sqrt{n/\ln n}2^n$ other edges. It

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is also worth noting that Erdős and Lovász guessed [7] that $m(n)$ is perhaps around $n2^n$.

The n-uniform, n-regular hypergraph is an interesting special case. The 2-colorability easily follows from the Lovász Local Lemma for $n \geq 9$, (see e.g., in [2]), for $n = 8$ it was proven by Alon and Bregman in [1], and finally Thomassen [13] showed it for $n \geq 4$.

Kostochka obtained the following lower bound on $m_k(n)$ in [8]. For every $k \in \mathbb{N}$, let $\epsilon(k) = \exp\{-4k^2\}$ and $r = \lfloor \log_2 k \rfloor$. Then for every $n > \exp\{2\epsilon^{-2}\}$, $m_k(n) \geq \epsilon(k)k^n(n/\ln n)^{r/(r+1)}$.

In this paper we use a different probability space which admits easier proof, though it gives weaker bounds. The main idea is to use greedy colorings on a random order of the vertices. Note that Radhakrisnan and Srinivasan [10] also used random vertex orderings after an initial random coloring. To generate a random order we let each vertex u pick a random real x_u uniformly and independently of each other from $[0, 1]$, and order the vertices according to these values. Equivalently, one can take a uniformly selected random element among the permutations of the vertex set, although the first form is better suited for the proof of Theorem 4.

In the next section we give a simple proof for the statement $m(n) > c_1\sqrt[4]{n}2^n$. The analysis of a random greedy algorithm also yields $m_k(n) > c_2k^{-1}n^{\frac{k-1}{2k}}2^n$. The constructions lead to a "characterization" of k-colorable hypergraphs which might be of interest by its own. We conclude the paper by a new proof for the 2-colorability of n-uniform, n-regular hypergraphs for $n \geq 8$.

2. Results

2.1. **2-coloring.** We define a random greedy coloring of a hypergraph $H = (V, E)$ as follows. Let σ be a uniformly picked random order of V. At the beginning all vertices are blue. In the i^{th} step we recolor the vertex $\sigma(i)$ to red if $\sigma(i)$ is the first element of an $A \in E$ according to the order σ.

Clearly, there are no completely blue edges in E at the end of the procedure. Let the number of completely red edges be X.

Claim 1. $\mathbb{E}X < 2\sqrt{\pi}e^{1/(6n)}n^{-\frac{1}{2}}2^{-2n}|E|(|E| - 1)$.

Proof of Claim 1. We say that $A \in E$ *precedes* $B \in E$ if the last vertex of A becomes red because it was the first element of B. If $X_{A,B}$ is the indicator variable of the event A precedes B then $X = \sum X_{A,B}$, where the summation runs over all ordered pairs of E. Hence

$$\mathbb{E}X = \sum \mathbb{E}X_{A,B} = \sum \Pr(A \text{ precedes } B) = \sum \frac{((n-1)!)^2}{(2n-1)!} = \sum \frac{2(n!)^2}{n(2n)!},$$

since A may precede B iff $A \cap B = \{x\}$ and x is the last element of A and the first element of B. Let us use the Stirling formula, i.e., $n! = \sqrt{2\pi n}(n/e)^n e^{\lambda_n}$, where $1/(12n+1) < \lambda_n < 1/(12n)$.

$$\mathbb{E}X = \sum \frac{2(n!)^2}{n(2n)!} \le |E||E-1|\frac{2\sqrt{\pi}e^{2\lambda_n - \lambda_{2n}}}{\sqrt{n}}2^{-2n} \le 2\sqrt{\pi}e^{1/(6n)}n^{-\frac{1}{2}}2^{-2n}|E|(|E|-1),$$

since $e^{2\lambda_n - \lambda_{2n}} < e^{1/(6n)}$. □

Corollary 1. *$m(n) > \frac{\sqrt{2}}{2}\pi^{-\frac{1}{4}}e^{-\frac{1}{12n}}\sqrt[4]{n}2^n$. That is $m(n) > 0.5268\sqrt[4]{n}2^n$, for $n \ge 3$.*

Proof. Just plug in $|E| = \frac{\sqrt{2}}{2}\pi^{-1/4}e^{-1/(12n)}\sqrt[4]{n}2^n$ into the formula of Claim 1. It gives $\mathbb{E}X < 1$, which means that there exists a good 2-coloring of (V, E). □

2.2. **k-coloring.** It is possible to get k-colorings by greedy algorithms for arbitrary $k \in \mathbb{N}$. Here greedy means that we color all the vertices with color 1, and in the i^{th} step we recolor the vertex $\sigma(i)$ if $\sigma(i)$ is a first element of an $A \in E$ according to the order σ. To recolor $\sigma(i)$ we use the smallest possible color that does not result in a monochromatic edge, otherwise we use the color k.

For an order σ of V, let $\{A_i\}_{i=1}^k$ be an *ordered k-chain* if $|A_i \cap A_{i+1}| = 1$, $A_i \cap A_j = \emptyset$ for $|i-j| > 1$ and $\sigma^{-1}(x) \le \sigma^{-1}(y)$ for all $x \in A_i$ and $y \in A_{i+1}$, $i = 1, \ldots, k-1$. If we have a fixed order σ, let $f(A)$ and $\ell(A)$ be the first and the last vertices of an edge A, respectively.

Lemma 2. *The hypergraph (V, E) is k-colorable if and only if there is an order σ of V containing no ordered k-chains. Moreover the greedy algorithm on (V, E) in this case provides a good k-coloring.*

Proof. For the "if" part let us color the vertices of V by the greedy algorithm in order σ. By the setup of the greedy algorithm, if there is a monochromatic edge $A_{k-1} \in E$ then its color can only be k. Now $\ell(A_{k-1})$ gets the color k since there is an edge, let us call it A_k, such that $\ell(A_{k-1}) = f(A_k)$. Similarly, $f(A_{k-1})$ is colored k, since there is an edge A_{k-2} such that $\ell(A_{k-2}) = f(A_{k-1})$, and all vertices of $A_{k-2} \setminus A_{k-1}$ are colored $k-1$. Taking $f(A_{k-2})$, we can get an $A_{k-3} \in E$ such that all vertices of $A_{k-3} \setminus A_{k-2}$ are colored $k-2$. By induction there is an $A_i \in E$ such that all vertices of $A_i \setminus A_{i+1}$ are colored $i+1$ if $i \ge 1$. But then $\{A_i\}_{i=1}^k$ is an ordered k-chain. The "only if" is trivial, given a good k-coloring let σ be an order induced by the colors, breaking the ties arbitrarily. □

Claim 2. *Let X be the number of k-chains in a random order of the n-uniform hypergraph (V, E), and $s = n - 1 > 0$. Then*

$$\mathbb{E}X < |E|^k \exp\left\{\frac{k}{12s} + 1\right\}(2\pi s)^{\frac{k-1}{2}}k^{-sk-1}s^{k-1}.$$

Proof of Claim 2. The proof is almost the same as that of Claim 1. Let $\mathcal{K}$ be the set of ordered k-tuples of A. For any $H \in \mathcal{K}$,

$$\Pr(H \text{ is a } k-\text{chain}) \leq \frac{\{(n-1)!\}^2\{(n-2)!\}^{k-2}}{\{(n-1)k+1\}!} = \frac{s!^k}{(sk+1)!s^{k-2}}.$$

Using the Stirling formula, with the bounds $e^{\lambda_s k} < e^{k/(12s)}$, $1 < e^{\lambda_{sk+1}}$ we get

$$\mathbb{E}X < |E|^k \exp\{\frac{k}{12s} + 1\}(2\pi s)^{\frac{k-1}{2}} k^{-sk-1} s^{k-1}.$$

$\square$

Corollary 3. *If $|E| \leq (2\pi e)^{-1/2} s^{\frac{k-1}{2k}} k^s$, then (V, E) is k-colorable. That is*

$$m_k(n) > (\sqrt{4\pi e}k)^{-1} n^{1/2-1/(2k)} k^n.$$

Proof of Corollary 3. If $|E| \leq (2\pi e)^{-1/2} s^{\frac{k-1}{2k}} k^s$ then there is an order σ of V such that (V, H) has no k-chain by Claim 2. Moreover $(\sqrt{4\pi e}k)^{-1} n^{1/2-1/(2k)} k^n < (2\pi e)^{-1/2} s^{\frac{k-1}{2k}} k^s$, and then (V, E) is k-colorable by Lemma 2. $\square$

Remarks. One can consider Lemma 2 from yet another point of view. Given a hypergraph (V, E) and a fixed order σ on its vertices, one may construct a directed graph $G_\sigma = (V(G_\sigma), E(G_\sigma))$. Let $v \in V(G_\sigma)$ iff $v = f(A)$ or $v = \ell(A)$ for some $A \in E$, and $(u, v) \in E(G_\sigma)$ iff there is an $A \in A$ such that $u = f(A)$ and $v = \ell(A)$. Obviously if for an order σ the graph G_σ has a good k-coloring then (V, E) is also k-colorable, and if (V, E) is k-colorable, then there exists an order σ such that G_σ is k-colorable. The non trivial part of Lemma 2 says that G_σ has a good k-coloring if it has no directed paths of length k. This is nothing else but a special case of an old result attributed to T. Gallai and B. Roy, that says if a directed graph G contains no paths of length k, then G is k-colorable, see chapter 9., problem 9 in [9].

2.3. **Sparse hypergraphs.** If a hypergraph (V, E) is *sparse*, that is each edge meets at most D other edges, then a good 2-coloring exists if $D \leq 0.17\sqrt{n/\ln n}2^n$ and n is big enough [10]. The direct use of the random orders and the Lovász Local Lemma gives

Theorem 4. *Let $H = (V, E)$ be an n-uniform hypergraph in which each edge meets at most D other edges. If $2e(2D^2 - D)((n-1)!)^2/(2n-1)! \leq 1$, then H is 2-colorable.*

Before the proof let us recall the Lovász Local Lemma. To spell it out we need a definition. If $A_1, ..., A_n$ are events of a probability space, then a *dependence graph* $G = (V, E)$ of these events is a graph having the following properties: $V = \{1, ..., n\}$, and each event A_i is mutually independent of the events $\{A_j : (i, j) \notin E\}$. Let $\deg_G(v)$ be a degree of a vertex v in G. For details see [2] and [7].

Lemma 5. *(Lovász Local Lemma)* [7] *Let $A_1, ..., A_n$ be events of a probability space, and G be a dependence graph of these events. If $\Pr(A_i) \leq p$ and $\deg_G(A_i) \leq d$ for all $1 \leq i \leq n$, and $ep(d+1) \leq 1$, then $\Pr(\cap_{i=1}^n \overline{A}_i) > 0$.*

Proof of Theorem 4. Let us consider the uniform random orders of V. For $A, B \in E$ let $\mathcal{A}_{AB}$ be the bad event that either A precedes B or B precedes A. Clearly, the event $\mathcal{A}_{AB}$ is mutually independent of all the other events $\mathcal{A}_{RS}$ when $(A \cup B) \cap (R \cup S) = \emptyset$. One checks that the number of intersecting unordered pairs $(R, S) \neq (A, B)$ that also intersects $A \cup B$ is not more than $2D^2 - D - 1$. Now the Lovász Local Lemma implies, there is an order σ containing no 2-chain, if

$$e\Pr(\mathcal{A}_{AB})(2D^2 - D) = 2e(2D^2 - D)((n-1)!)^2/(2n-1)! < 1.$$

This inequality holds by assumption, so H is 2-colorable by Lemma 2. □

Remark. A quick asymptotic of Theorem 4 gives that such hypergraphs are 2-colorable if $D < 0.23\sqrt[4]{n}2^n$. This result is asymptotically weaker than the former $0.17\sqrt{n/\ln n}2^n$ bound, but Theorem 4 has better constants and works for *all* $n > 1$. It already implies the known results of the values for which an n-uniform, n-regular hypergraph is 2-colorable. Note that this follows from the Lovász Local Lemma easily if $n \geq 9$, while for the case $n = 8$, see the paper of Alon and Bregman, [1].

Corollary 6. *Every n-uniform, n-regular hypergraph is 2-colorable, for $n \geq 8$.*

Proof of Corollary 6. First we show a sharp bound on Δ_n, the number of intersecting unordered pairs $(R, S) \neq (A, B)$ that also intersects $A \cup B$. Observe that the number of pairs intersecting with the fixed (A, B) is maximum when (V, E) is *almost disjoint*, i.e., for every $R, S \in E$ we have $|R \cap S| \leq 1$ if $R \neq S$.

From the n-regularity we have

$$\Delta_n \leq 2(n-1)^4 + 2(n-1)\binom{n-1}{2} + \binom{n-2}{2} + 2(n-2).$$

Following the proof of Theorem 4, the Lovász Local Lemma implies that if

$$f(n) := 2e(\Delta_n + 1)((n-1)!)^2/(2n-1)! < 1,$$

then an n-uniform, n-regular hypergraph (V, E) is 2-colorable. Since $f(8) \leq 0.604$, Corollary 6 follows. □

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