The accelerated k-in-a-row game *

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Abstract

We investigate two versions of the well-known k-in-a-row game. While in the most intriguing $k = 5$ case the outcome of the game has been recently settled, very little is known about what happens when the rules are changed. A natural modification is that the players take more than one square of the board per move in order to speed up the game. Our main goal is to improve the quadratic bound on the error term, given by Csirmaz in [6], to a logarithmic one for the accelerated k-in-a-row. The other issue is the extrem sensitivity of k-in-a-row under biased rules. Beck proposed in [4] that a player may trade some of his freedom of choice for the right of taking more squares than his opponent. We prove logarithmic bounds on the error term in that case, too.

Keywords: positional games, derandomization, k-in-a-row.

1 Introduction and Results

The game *k-in-a-row* ($k \in N$) is played by two players (I and II) on the infinite (chess)board. They alternately put their own marks to previously unmarked squares, and whoever gets k-consecutive marks first (horizontally, vertically or diagonally) of his own, wins. A number of results are known for this game. The *strategy stealing argument* shows the advantage of the first player, i.e. assuming perfect play from both players, either I wins or it is a draw game. These are the only possible outcomes for any $k \in N$ (see in [9]). Furthermore, I wins if $k \leq 5$, see in [1], while II can prevent I from winning if $k \geq 7$ (see in [8]). Even more is true when $k \geq 7$: I cannot achieve k-consecutive marks even if the game continues indefinitely (no matter if II completes some "k in a row"). This version is no longer symmetric, and it is much harder for II to prevent I from winning this game. Such an asymmetric version of a positional game is called the *Maker-Breaker version.* The advantage of this form is that it still gives meaningful bounds and makes the use of combinatorial tools possible.

We shall investigate a more general situation, which first occurred in [6]. A family of new games, what we shall call $A_k(n,m)$, where A stands for both *asymmetric* and *accelerated*, can be defined with the following modification: In every step of the game I and II mark n and m previously unmarked squares, respectively. I wins the game by marking k consecutive squares (horizontally, vertically or diagonally).

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It was shown in [6] that I wins the game $A_k(n,m)$ for every k if $n > m$, and II can prevent I from winning it if $8n \le m$ and $n \le k$. Indeed, the really interesting case is when $n = m$, i.e. the *diagonal case.* I wins the game $A_k(n,n)$ if $k \le n$, therefore it is natural to write k in the form $k = n + f(n)$, and to look for criterions of the outcome in terms of $f(n)$. The main result in [6] is that if $f(n) = cn^2$ for an appropriate constant c, then II has a tie. Using a Hales-Jewett type pairing strategy a linear bound was proved on $f(n)$ in [10]. Here we improve on this bound even more.

Theorem 1 *If $f(n) \ge 80\log_2 n + 160$ and $n \ge 1000$, then II can tie $A_{n+f(n)}(n,n)$.*

Probably I can win when $f(n) = \Omega(\log n)$, but we have only a slightly weaker bound.

Theorem 2 *If $f(n) \le \log_2 n / \log_2 \log_2 n - 1$, then I wins the game $A_{n+f(n)}(n,n)$.*

As we have mentioned, I can win the game $A_k(n,m)$ for $n > m$ easily. However, if a natural restriction is imposed on the moves of I, then II can still tie the game. We call the game $A_k(n,m)$ to be $A_k(n,m;d)$ if within a step I's marks must lie in a circle of diameter d. (The squares of the board are the copies of the unit square, and $A_k(n,m) = A_k(n,m;\infty)$.) This approach, and results on the game $A_k(2,1;d)$ first came up in [4]. Since [4] has never been in print, hereby, with the permission of its author, we explore this direction, and give a generalization of his results.

Theorem 3 *If $f(n) \ge 240\log_2 d + 480$ and $d \ge 4000$, then II can tie $A_{n+f(n)}(n,n/2;d)$.*

As we shall see from the proof, fixing $m = n/2$ in Theorem 3 is rather arbitrary, any $m = \varepsilon n$, $\varepsilon > 0$ would work. As a matter of convenience, we have already assumed implicitly that n is even. Theorem 3 tells us that II ties if k exceeds n by a logarithmic function of d. On the other hand, I can win for $k = n + c\log_2 d$, where $c > 0$, if d is much bigger than n.

Theorem 4 *If $f(n) \le \log_2(d/n)/\log_2 5$, then I wins the game $A_{n+f(n)}(n,n/2;d)$.*

2 Preliminary

Before starting the proofs we need some definitions and to recall former results from [2], [4] and [7].
A pair of (X,H) is called a *hypergraph* if $H \subset 2^X$. If (X,H) is a hypergraph, then a $(p,q,H)-$ *game* (or simply hypergraph game) is a game in which I selects p and II select q previously unselected elements of X. The first, who takes all elements of an $A \in H$, wins. A (p,q,H)-game has a so-called *Maker-Breaker version* in which I wins taking an edges of the hypergraph any time.

These games are usually hard and always "hot" in the sense of [5], hence the otherwise effective "chilling" does not help. What works is the method of *weight functions*, see in [2], [4] and [7]. The following theorem may be called as the Fundamental Theorem of hypergraph games.

Theorem 5 ([2]) *If* $\sum_{A\in H}(1+q)^{-\frac{|A|}{p}} < \frac{1}{1+q}$ *, then* II *can prevent* I *from winning the Maker-Breaker* $(p,q,H)-$ *game.*

Corollary 1 *Suppose* H *is* k*-regular. If* $2|H| < 2^{k/p}$ *then* II *can force a draw in the game* $(p,1,H)$.

While it is an immediate consequence of Theorem 5, for the sake of completeness and for future references, we prove it.

Proof. For any $A \in H$ let $A_k(I)$ and $A_k(II)$ be the number of elements in A, after I's kth move, selected by I and II, respectively. Now, for an $A \in H$

$$w_k(A) = \begin{cases} \lambda^{A_k(I)} \text{ if } A_k(II) = 0 \\ 0 \text{ otherwise,} \end{cases}$$

where $\lambda > 0$, and for any $x \in X$ let $w_k(x) = \sum_{x\in A} w_k(A)$. The numbers $w_k(A)$ and $w_k(x)$ are called the *weight* of A and x (in the kth step), respectively. When it does not cause confusion we may suppress the lower index.

Now selecting an element in the kth step II uses the *greedy* algorithm, i.e. chooses an unselected element $y^k \in X$ of maximum weight. Let $x_1^{k+1}, ..., x_p^{k+1}$ be the elements selected by I in the $(k+1)$st step and $w_k = \sum_{A\in H} w_k(A)$ be the *total sum* or *potential.* For $k \geq 0$, following inequality holds for the potential:

$$w_k - w_k(y^k) + (\lambda^p - 1)w_k(y^k) \geq w_{k+1}.$$

Indeed, w_k decreases by $w_k(y^k)$ upon selecting y^k. The elements selected by I in the $(k+1)$st step cause the biggest increase if $w_k(x_l^{k+1})$ is maximal for $1 \leq l \leq p$, and for all A such that $w_k(A) \neq 0$ we have $x_l^{k+1} \in A$ iff $x_m^{k+1} \in A$, $1 \leq l,\ m \leq p$. Since the increase in this case is just $(\lambda^p - 1)w_k(y^k)$, the inequality is proved. Setting $\lambda = 2^{1/p}$, we get $w_k \geq w_{k+1}$, $k \geq 0$, which justifies that w_k is called potential.
Particularly $w_1 \leq (\lambda^p - 1)|H| + |H| \leq 2|H|$. Since $q = 1$ and the elements of H are the same size, the inequality $\sum_{A\in H}(1+q)^{-|A|/p} < 1/(1+q)$ leads to the inequality $2|H| < 2^{|A|/p}$. Let us suppose that I wins the game in the kth step. This would imply $w_k \geq \lambda^{|A|} = 2^{|A|/p}$, which contradicts the monotonicity of the potential.□

An edge $A \in H$ is *active* if II has not taken any of its elements. Since $w_k \leq w_1 \leq 2|H|$ for all k, we have a bound on the "fill-in" of an active edge. Note, that this bound holds for the non-uniform hypergraphs, too.

Corollary 2 *Playing the Maker-Breaker* $(p,1,H)$ *game,* II *may arrange that whenever* $A_k(II) = 0$, *then* $A_k(I) \leq p + p\log_2|H|$.

Proof. Just take the logarithm of the inequality $\lambda^{A_k(I)} = w_k(A) \leq w_k \leq w_1 \leq 2|H|$ that holds for any active edge $A \in H$. □

However, these corollaries cannot be used directly in our case because of several reasons. One of these is that the value of the weight function grows enormously during a step of I, since $p = n$ here. We shall play *auxiliary games* for which the objectives are different from the original one, but the appropriate play of these games results in the desired outcome of the original game.

We may state a slight generalization of the Corollary 2, which turns out to be very handy. Suppose we are given a hypergraph (X, H) and an abstract *blocking rule* $\mathcal{B}$. I and II select the elements of X (just as before in the (p, q, H)-game), and I wins if he gets all elements of an $A \in H$, which is unblocked by the other player. Informally, if given a certain position of the game, we have a subset of the edges which are "blocked."

Formally the blocking rule $\mathcal{B}$ is a function of the hypergraph edges and the already played part of the game,

$$\mathcal{B} : H \times 2^X \times 2^X \longmapsto 2^{2^X}.$$

Let X_I (X_{II}) be the subset selected by I (II) from X at a moment, then an $A \in H$ is blocked by II (I) if there exists a set $Y \in \mathcal{B}(A, X_I, X_{II})$ such that $Y \subset X_{II}$ (X_I). (In the (p, q, H)-game $\mathcal{B}(A, X_I, X_{II}) = \{\{x\} : x \in A\}$.) An $x \in X$ is *blocked* if for all $A \in H$, containing x, is blocked.
Having these notions, a weight function can be defined, much like before. For an $A \in H$,

$$w(A) = \begin{cases} \lambda^{A(I)} \text{ if } A \text{ is not blocked by } II \\ 0 \text{ otherwise} \end{cases}$$

is the weight of A, for an $x \in X$, $w(x) = \sum_{x \in A} w(A)$ is the weight of x and $w = \sum_{A \in H} w(A)$ is the potential. The lower-index has already been omitted because we want to modify the previous algorithm.

Our goal is to establish a bound on the fill-in of an unblocked edge A, similar to that of Corollary 2. If both players mark n-n elements in each move, then this bound is poor when I has just finished his move, but may be better after II's move.

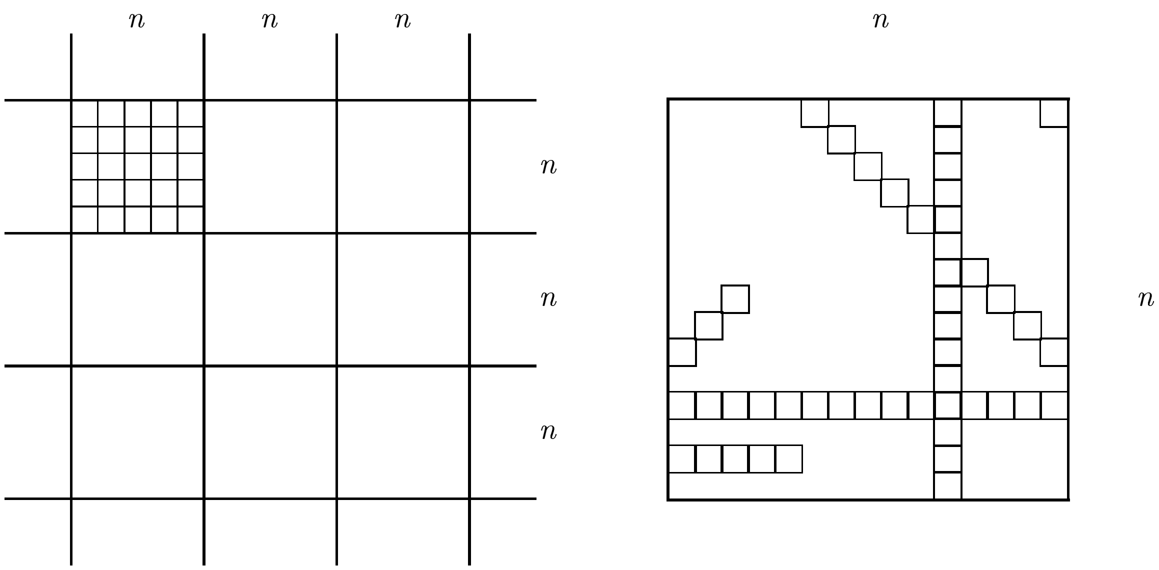
Note that if II can block *all* edges containing an element z, then its effect on the weight function is just the same as if II would have taken z. Such a z is *neutralized*, and the game is a $\mathcal{B}(p, 1, H)$-*game* if the structure of H allows II to neutralize any $z \in X$ at any time, by using at most p of his marks. When II is to move, he partitions I's previous move (a sequence of marks) into subsequences of lenght p, computes the weight function after the first subsequence, and neutralizes the element z of maximum weight. Then he considers the following subsequence, and so on. Doing so, *after* his move he has the same bound on the value of the weight function as if he were playing a $(p, 1, H)$-game. Consequently we have:

Lemma 1 *In a $\mathcal{B}(p, 1, H)$ – game after II's kth move the potential $w_k \leq 2|H|$, consequently if $A(I) = i$ and A is unblocked by II then $i \leq p + p \log_2 |H|$ for every $A \in H$ and $k \in \boldsymbol{N}$.*

3 Proof of Theorem 1

The basic idea of the proof is to identify the winning sets of the game with an appropriate $\mathcal{B}(p, 1, H)$-game and exploit the bound of Lemma 1. A number of technical problems must be tackled; the first one is that we have an infinite board.

We divide up the infinite board into n by n square sub-boards. A *slice* S of a sub-board is a set of consecutive (horizontally, vertically or diagonally) squares such that at least one end of the slice S touches the border of the sub-board.



subdivision of the board

slices on sub-board B

figure 1

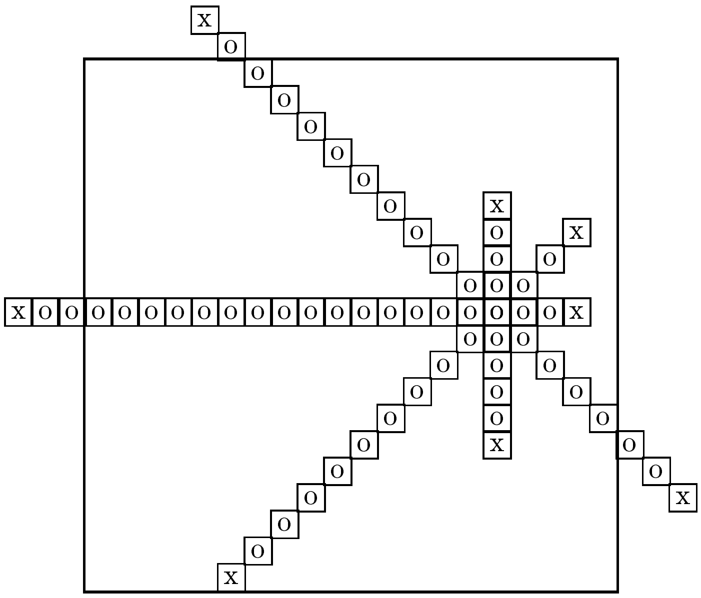
Obviously, there are less than $8n^2$ slices on a sub-board. However, this decomposition itself is not enough, since it leads only to an (n, n, H)-game on the sub-boards, where H is the set of slices of a sub-board, i.e. $|H| < 8n^2$. The other problem is that a sequence of I's marks, starting on a sub-board B_1, may be continued on a neighboring sub-board B_2. A natural way to overcome both of these difficulties is to utilize a flexible blocking rule. (Similar problems and ideas appear in [12].) The slice S is *blocked* (by II) if either II marked a square of S or II marked squares q and $\bar{q}$ on both sides of the line of S and there are no unmarked squares between q and $\bar{q}$.

More formally, let X_I and X_{II} be the squares marked by I and II, and let us call the squares between q and $\bar{q}$ $segment(q, \bar{q})$ if the squares q and $\bar{q}$ are elements of a set of consecutive squares. With this notation

$$\mathcal{B}(S, X_I, X_{II}) = \{q : q \in S\} \cup \{\{q, \bar{q}\} : S \subset segment(q, \bar{q}) \subset X_I \cup X_{II}\}.$$

Computing the weight of a slice S, we take $\lambda = 2^{1/8}$. For the efficient use of Lemma 1, we need a decomposition, which enables II, so to say, *to insert* his marks into the length n sequence of I's marks. Having this decomposition, II uses a strategy that we call *very greedy strategy.*

In any step I marks $8j+i$ ($0 \le j \le \lfloor n/8 \rfloor$, $0 \le i \le 7$) squares on an n by n sub-board B. Let us index $8j$ of these squares by the numbers $1, 2, \cdots, 8j$ and call them *type*-8 squares, and call the remaining i squares of *type*-7. After a step of I, II looks up the sub-boards where $j \neq 0$ or $i \neq 0$, one by one. If for a sub-board B $j \neq 0$ then II tries to encounter the type-8 squares first, doing it in j sub-stage. At the beginning of the sth sub-stage ($s \le j$) II pretends that I marked only the squares indexed by $1, \cdots, 8s$, and leaves out the squares indexed by $8s+1, \cdots, 8j$ and the type-7 squares from the computation of the weight function. Then based on this partial weight function, II selects a square z of maximum weight, *marked or not by I*, and blocks it in the following way:
1. If z is unmarked then II marks it.
2. If z has been marked by I then II looks up the first unmarked squares $z_1, ..., z_8$ from z to each of the eight directions. If II has already marked a square between z and z_l ($1 \le l \le 8$) then does nothing, otherwise marks z_l.



II's marks are represented by crosses. The center square is blocked.

figure 2

Finally, to encounter the i type-7 squares, II marks i unmarked squares, each of these of maximum weight at the moment.
We shall see similar situation in Theorem 3, too. For the sake of a unified treatment we finish the proof of Theorem 1 with a slightly stronger statement. Let us suppose that I and II, marking the squares of the infinite board, play an arbitrary game, and II, using a system of $n \times n$ square sub-boards, the slices of these sub-boards and

the very greedy strategy, manages to turn the game into a $\mathcal{B}(p, 1, H)$-game on every sub-board. With this setting, we have the following lemma.

Lemma 2 *If $n/p \geq 10\log_2(4n)$ then a set of consecutive (horizontally, vertically or diagonally) squares, marked by I, has less than $n + 20p + 10p\log_2 n$ elements.*

Proof of Lemma 2. We shall use an induction on the number of steps. It is trivial in the first step, since I marked only n squares. Now let us assume Lemma 2 holds up to the first i step and after I's $(i+1)$st move there exists a set E of consecutive squares, such that $|E| = n + 20p + 10p\log_2 n$, and all elements of E is marked by I. Since I could mark only n of these in the last step, at least $20p + 10p\log_2 n$ squares had been marked before. On the other hand, at least one element of E was marked by I at the last step, otherwise the induction hypothesis would be violated. It follows that at the end of the ith step no slice $S \subset E$ had been blocked by II. The assumption $n/p \geq 10\log_2(4n)$ means $n \geq 20p + 10p\log_2 n$, which implies that E is in the union of at most five sub-boards.

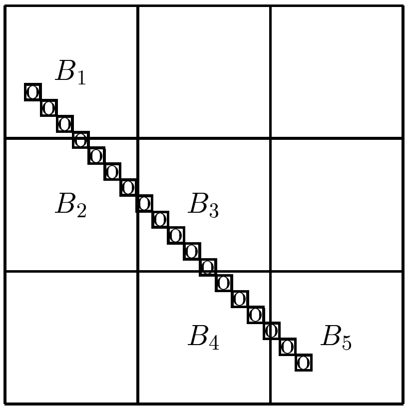


figure 3

Denoting these sub-boards by B_l, $1 \leq l \leq 5$, E is the union of at most five slices $S_l = B_l \cap E$. By the pigeon-hole principle there must be an l, $1 \leq l \leq 5$, such that S_l has at least $4p + 2p\log_2 n$ elements, marked by I at the end of the ith step. On the other hand, S_l and all the other slices, which are part of S_l, are unblocked, and the sum of their weights is strictly greater than

$$\lambda^{4p+2p\log_2 n} = 2|H|.$$

But this contradicts Lemma 1, so Lemma 2 is proved. $\square$

Finally, Lemma 2 yields Theorem 1, since in that case $p = 8$. $\square$

4 Proof of Theorem 2

First we give the sketch of I's strategy. He shall use the division of the board defined in the proof of Theorem 1. Then he selects N ($N >> n$, its value will be specified later) sub-boards of the infinite board such that their mutual distances are at least $4n$. (This way there will be no interference among the marks of different sub-boards.) The actual play consists of two phases; the first phase ends if after II's turn there is a sub-board B such that one of the rows or columns of B contains $y = \log n / \log\log n - 1$ marks of I, and none of II. Then, in the second phase I simply fills the gaps marking n squares of that row/column, and wins the game.

To finish the proof, we have to show that I has a strategy that results in the above described ending of the first phase. In each turn I visits n/y sub-boards, marking y squares on each. However, after II's move, only those sub-boards remain *active* which received not more marks of II than that of I. In one sub-phase I visits all the active sub-boards, of which at least $\alpha = 1/(y+1)$ percentage will be active after II moves. We shall see that I needs to visit an active sub-board less than n times (i.e. there are no more than n sub-phases), so he will not run out of active sub-boards if $N > \alpha^{-n}$.

Now we describe I's strategy on a sub-board B that stays active through the whole phase. (Note that within B, I can treat the game as a $(y,\, y,\, H)$-game, that is taking turns, I and II mark y squares in each steps.) Again, a row or column is *active* if it does not contain a mark of II neither in B nor on its neighbors. Now the general step is the following: I looks up the first active row, and marks its first y squares that are in active columns. Of course, II has to close this row from both left and right, otherwise I would win immediately. Because of this, at least two columns, in which I has marked a square right now, will be active after II's move. Then I moves to the next available active row, and does the same. In the response II may mark such a square that closes the row, filled by I, from the *left* and lies one of the active columns that received one of I's mark in the previous sub-phase. Still, after j sub-phases, there will be j active columns containing one mark of I. Note that if I follows this strategy through $j = n/2y$ sub-phases, then at least $n/2$ rows of B remain active.

That was the first stage of I's strategy, and the subsequent stages are very similar. In the second stage I considers only those active columns in which he has marked a square already. Following the same pattern as before, he ensures that at the end of the second stage there will be $n/(2y^2)$ active columns such that he has marked two squares of each of these columns. (And the number of active rows in B is still more than $n/2$.) An induction shows that after the ith stage there will be $n/(2y^i)$ active columns, each containing i squares marked by I, and the number of active rows is still not less than $n/2$. Since $n/2y^y \geq 1$, at the end of the yth stage the first phase is also over, and I wins the game. $\square$

5 Proof of Theorem 3

The proof is based on the same strategy as the proof of Theorem 1. First we divide up the infinite board into d by d square sub-boards. On a sub-board B a slice S and the blocking rule $\mathcal{B}$ are defined as before. Here $\lambda = 2^{1/24}$.

Now, because II has only $n/2$ marks per step, we have to change the distribution of them. When I marks $16j + i$ $(0 \leq j \leq \lfloor n/16 \rfloor,\ 0 \leq i \leq 15)$ squares on a sub-board B we index $16j$ of these squares by the numbers $1, ..., 16j$ and call them *type* $- 16$

squares, while the rest are the $type-15$ squares. The sequence of squares of type-16 can be handled the same way as before, since II may use $8j$ marks for this on B. Reducing the to a $\mathcal{B}(p,1,H)$-game the real difficulty comes with the squares of type-15, since two times as many sub-boards may receive type-15 squares than the number of II's available marks for the encounter. Here the diameter limitation helps. (A similar idea works in [11].)

Lemma 3 *II can play a $\mathcal{B}(24,1,H)-game$ on every sub-board B where $H \cong H(B) = \{$ S: S is a slice of B $\}$ and the blocking is the same as before.*

Proof of Lemma 3. As earlier, encountering the type-16 and type-15 squares II places marks according to the very greedy strategy; the only problem is to select the sub-board of response. On a sub-board B, II encounters type-16 squares and uses $\lfloor i/2 \rfloor$ to respond to the i type-15 squares. If there are sub-boards where the number of type-15 squares is one then II has left at least one mark since we assumed that n is even. He selects one among these sub-boards, using a rule which guarantees if a sub-board B received single type-15 squares nine times between the steps s_1 and s_2 then B was selected by II at least once in this interval of steps. Assuming this rule, Lemma 3 follows, since the longest unencountered sequence of I's marks may consist of eight single type-15 squares and sixteen type-16 squares. In other words, because of the decomposition via the the type-16 and type-15 squares, and a rule of the claimed property, the very greedy strategy yields the blocking of at least every 24th square of maximum weight.

Because of the diameter limitation, within one step I's marks are in a subset of the infinite board that consists of nine sub-boards. These are B_l $(1 \le l \le 9)$ and $D = \{B_l : 1 \le l \le 9\}$.

Let us denote the eight sub-board surrounding a sub-board B by $N(B)$. In order to define the selection rule for II, we introduce a relation $\mathcal{O}$ on the set of sub-boards. At the beginning of the game the relation is empty. We say B_s *owes* B_r if $(B_s, B_r) \in \mathcal{O}$. Now II selects a $B^* \in D$ for which B^* received a single type-15 square and B^* does not owe any sub-board $B \in D \cap N(B^*)$.

Finally II updates the relation $\mathcal{O}$. B^* owes all $B \in D \cap N(B^*)$ and no $B \in D \cap N(B^*)$ owes B^* in the updated relation. Now, if a sub-board B, having a single type-15 square, was not selected then a B^* was. B^* owes B, and cannot be selected again until B is selected. Since $|N(B)| = 8$, the Lemma 3 is proved. □

As Lemma 3 shows, the conditions of Lemma 2 hold, which gives Theorem 3. □

6 Proof of Theorem 4

As we shall see I needs only one type of winning direction (e.g. the vertical) and a rectangular sub-board B of height $\log_5(d/n) + n$ and width d to win the game. Indexing the squares from the upper-left corner we may refer to B as a $\log_5(d/n) + n$ by d matrix. When one of the players puts a mark to the square labeled by (i,j) we simply say that I or II selected b_{ij}. Let the jth column of B be *free* if II has not taken any b_{ij}, $1 \le i \le \log_5(d/n) + n$. Similarly, an unselected b_{ij} is *free* if the jth column is free. Now I first plays the following *growing* strategy. He decomposes the game into stages $1, ..., \log_5(d/n)$, and selects only among the elements of the ith row in the ith step. I simply takes the elements of the ith row, which are in free columns, and completes the ith stage and starts the $(i+1)$st stage when there are no more unselected free elements in the ith row. It is clear that a stage is never longer than d/n steps, i.e. a stage is always finite.

Lemma 4 *During the execution of the previous algorithm there are free columns at the ith stage if $i \leq \log_5(d/n)$.*

Proof of Lemma 4. Let us denote the number of free columns before the ith stage by F_i, and the number of elements selected by I and II by $E_i(I)$ and $E_i(II)$, respectively. Since I always selects a free element (from a free column) $F_{i+1} \geq E_i(I) - E_i(II)$. Moreover, it is also true that $E_i(I) + E_i(II) \geq F_i$, since I occupies every free element in the ith row.
We claim if $F_i \geq 5n$, then $2E_i(I)/3 \geq E_i(II)$. Indeed, I selects roughly twice as many elements than II, if a stage is long enough. For the short stages which contain only a few steps the worst case is when a stage starts and ends with a step of II. (More precisely, the previous stage ends when I used up almost all of his n marks, and the current stage reaches the completion at the beginning of an I's step.)
Since $F_i \geq 5n$ even in the worst case the stage consists of at least three complete steps of I, and at most five steps of II, which gives the claim.

Comparing the former inequalities, we get $F_{i+1} \geq F_i/5$, provided that $F_i \geq 5n$. Since $F_1 = d$ and $F_1 \geq F_2 \geq ... \geq F_i$ if i is such that $5^{1-i}d \geq 5n$, then $F_i \geq 5n > 0$. But this is exactly what we wanted, $F_{i+1} > 0$ if $i \leq \log_5 d/n$. ($F_{i+1} > 0$, because I can play one more stage, having $F_i \geq 5n$.) □

Obviously, a free column j contains i elements, selected by I, by the end of the ith stage. (Namely, $b_{1j}, ..., b_{ij}$ are those elements.) Indeed, the kth stage ($1 \leq k \leq i$) was completed only after b_{kj} had been selected.

Now Theorem 4 easily follows from Lemma 4. I, after having played the growing strategy for $i = \log_5(d/n)$ stages, selects a free column j, and puts marks onto the squares corresponding to $b_{i+1,j}, b_{i+2,j}, ..., b_{i+n,j}$. □

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