Generalized Harary Games *

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Abstract

There are a number of positional games known on the infinite chessboard. One of the most studied is the 5-*in-a-row*, whose rules are almost identical to the ancient Japanese *Go-Moku*. Along this line Harary asked if a player can achieve a translated copy of a given polymino P when the two players alternately take the squares of the board. Here we pose his question for general subsets of the board, and give a condition under which a draw is possible. Since a drawing strategy corresponds to a good 2-coloration of the underlying hypergraph, our result can be viewed as a derandomization of the Lovász Local Lemma.

1 Introduction and Results

Frank Harary proposed the following game on the infinite chessboard (two dimensional lattice) which resembles both the k-in-a-row, and the Hex (see [5], [8]). Let us recall that a *polymino* is a set of connected squares of the chessboard. Given a polymino P, the players, I and II take one square of the chessboard at each turn. I tries to take a translated copy of P, while II's goal is to prevent I from doing this. A polymino P is a *winner* if I has a winning strategy, otherwise P is a *loser*. Andreas Blass found most of the minimal loser polyminoes, using Hales-Jewett-type of pairings (see [5]). The opposite task, that is to decide about the winners, was carried out exhibiting some sequences of winning moves. Practically all of the winner polyminoes are known, there are at most twelve of them. The status of the largest one, called *snaky*, is still unsettled. Although it seems to be a winner, no one has found a convincing proof yet (see [5]).
In this paper we are interested in a more general situation:

1. II has to prevent I from taking not only one, but several other polyminoes,
2. P is not necessary a polymino (i.e. connected set of squares).

The methods, used in the solution of the original problem, break down hopelessly in the generalized cases. (Especially, when both modifications are considered.) If we

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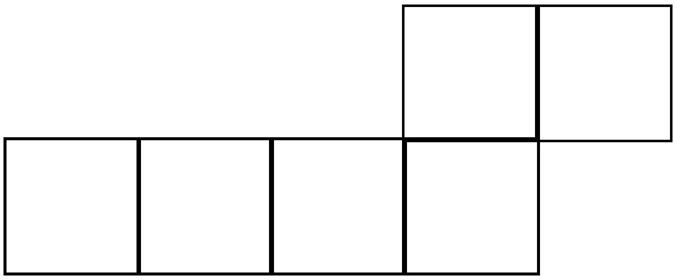

Figure 1: The polymino called snaky

have some additional properties of the winning patterns, then the weight function technique still can provide an answer. First we need to formalize the notion of a *hypergraph game.*

Definitions:
An (X, H) pair is a *hypergraph* if $H \subset 2^X$. Given a hypergraph (X, H) the players I and II can play the following game that we call (p, q, H) *hypergraph game* (or shortly (p, q, H) *game*):
I and II select p and q unselected elements of X in each turn, respectively. The first, who selects all elements of an $A \in H$, wins.
For $1 \leq i \leq k$ let P_i be a set of squares of the infinite chessboard, $n_i = |P_i|$ the number of elements in P_i, and $d(P_i)$ the diameter of P_i in Euclidean norm. Let $A(p, q; \mathcal{P})$ be an (p, q, H) game, where X is the infinite board and H consists of the translated copies of $P_1, ..., P_k$. Furthermore, set $n = min\{n_i : 1 \leq i \leq k\}$ and $d = max\{d(P_i) : 1 \leq i \leq k\}$.

Theorem 1 *If $n \geq 50 \log_2 d + 25 \log_2 k + 25$, then II can prevent I from winning the game $A(1, 1; \mathcal{P})$.*

It is natural to ask what happens in general, that is if I wins the game $A(p, q; \mathcal{P})$, or it is a draw. A similar argument like in [10] would show that I wins every $A(p, q; \mathcal{P})$ if $p > q$. (Omitting the details, we give just a sketch of I's strategy: Take squares far from each other for some turn. In the subsequent turns neglect those which are "too close" to squares taken by II. Since $p > q$, I can build up an arbitrary pattern.) Hence the only open cases where $p \leq q$. The most intriguing case the $p = q = 1$, although we believe that dropping the diameter restriction does not really help I, which is spelled out in Conjecture 1.

Conjecture 1 *There exists a function $f(k)$ depending only on k such that, if $n \geq f(k)$, then II can prevent I from winning the game $A(1, 1; \mathcal{P})$.*

2 Weight Functions

In this section we shall recall the most useful method in the theory of hypergraph games, the method of *weight functions*. It is impossible to trace back when it did first appear, in some sense the idea is as old as the exponential function. It has surfaced in the recreational mathematics several times, eventually P. Erdős and J. L. Selfridge put it to its proper place in mathematics (see [7]). Later a large number of applications were found, both in the theory of games and in other parts of discrete mathematics (see for example [1, 2, 3, 4]). The following result may be called as the "Fundamental Theorem of Hypergraph Games"; for the case $p = q = 1$ it was proved in [7], the general form is from [2].

Theorem 2 *[2, 7] II can prevent I from winning the (p, q, H)-game if*

$$\sum_{A \in H} (1+q)^{-\frac{|A|}{p}} < \frac{1}{1+q}.$$

This theorem cannot be used in our case directly, since $\sum_{A \in H} 2^{-|A|}$ is *not* finite. Yet another difference is that it does not harm the player II if he gets extra elements from X. Intuitively it is clear that II is better off if in some turns he can take more than q elements of X (even if he receives the extra elements randomly). We call a hypergraph game *relaxed* (p, q, H) game if at each turn I takes at most p, while II takes at least q elements of X. Theorem 2 holds for the relaxed (p, q, H) games, too. For the sake of compactness, we repeat Beck's proof from [2], getting Lemma 1.

Lemma 1 *[2] II can prevent I from winning the relaxed $(p, 1, H)$-game if*

$$\sum_{A \in H} 2^{-\frac{|A|}{p}} < \frac{1}{2}.$$

Proof of Lemma 1.
For any $A \in H$ let $A_k(I)$ and $A_k(II)$ be the number of elements in A, after I's k^{th} move, selected by I and II, respectively. Furthermore

$$w_k(A) = \begin{cases} \lambda^{-|A|+A_k(I)} \text{ if } A_k(II) = 0 \\ 0 \text{ otherwise} \end{cases},$$

where $\lambda > 0$ and for any $x \in X$

$$w_k(x) = \sum_{x \in A,\ A \in H} w_k(A).$$

The numbers $w_k(A)$ and $w_k(x)$ are called the *weight* of A and x (in the k^{th} step), respectively.
For selecting an element in the k^{th} step II uses the *greedy* algorithm, i.e. he chooses an unselected element $y^k \in X$ of maximum weight. Let $x_1^{k+1}, ..., x_p^{k+1}$ be the elements selected by I in the $(k+1)^{st}$ step and let

$$w_k = \sum_{A \in H} w_k(A)$$

be the *total sum* or *potential.* The following inequality holds for the potential:

$$w_k - w_k(y^k) + (\lambda^p - 1) w_k(y^k) \geq w_{k+1}$$

if $k \geq 0$. Indeed, w_k decreases by $w_k(y^k)$ upon selecting y^k. On the other hand, it is easy to see that the increase of the potential, caused by I's newly selected elements, is the greatest in the case where:

1. $w_k(x_l^{k+1})$ is maximal for $1 \leq l \leq p$
and
2. if $w_k(A) \neq 0$ $(A \in H)$, then $x_l^{k+1} \in A$ iff $x_m^{(k+1)} \in A$, $1 \leq l$ and $m \leq p$.

But the increase in this case is just $(\lambda^p - 1)w_k(y^k)$, therefore the inequality is proved. Setting $\lambda = 2^{1/p}$, we get

$$w_k \geq w_{k+1},$$

for $k \geq 0$, which justifies that w_k is called potential.
Particularly

$$w_1 \leq 2(\lambda^p) \sum_{A \in H} 2^{-|A|} < 1.$$

Let us suppose that I wins the game in the k^{th} step, occupying the set A. This would imply

$$w_k \geq \lambda^{-|A|+A_k(I)} = 0,$$

which contradicts the monotonicity of the potential. □

Remarks.
1. Intuitively, the potential measures the overall danger that the vertices of the elements of H are being selected by I during the game. Most often it is done by choosing an appropriate exponential function, and this exponentiality is to which one can attribute the power of the weight function method. Practically speaking, one may expect reasonable theorems via weight functions for a family of hypergraphs

$$\mathcal{F} = \{(X, H) : \gamma \in \Gamma\}$$

if there exists a polynomial p, such that

$$|H_\gamma| \leq p(|X_\gamma|)$$

for all $\gamma \in \Gamma$. As we shall see, the special structure of the hypergraphs can also help, even when $|H_\gamma| = \infty$.
2. There is a deep connection between the random 2-colorings of a hypergraph (X, H) and the $(1, 1, H)$-game. The inequality

$$\sum_{A \in H} 2^{-|A|} < \frac{1}{2}$$

says that the expected number of monochromatic elements of H is less than 1, i.e. there is a good 2-coloring. From this point of view the weight function technique is nothing else but the *derandomization* of the well known first moment method. The conditional probabilities for certain events can also be interpreted as weight function values (see [1]). This method makes it possible to turn probabilistic algorithms into effective, polynomial time deterministic ones (see [4]). On the other hand, if the expected number of monochromatic sets is "small", then II might achieve a draw in the corresponding hypergraph game.

3 Proof of Theorem 1

First we cut up the board into $d \times d$ squares, and call the set of these squares $\mathcal{S}$. For an $S \in \mathcal{S}$ let $\bar{S}$ be the union of S and the other eight $d \times d$ squares surrounding S. We shall refer to $\bar{S}$'s as the sub-boards. For an $\bar{S}$ the game $(\bar{S}, H(\bar{S}))$ is the hypergraph game, where the winning sets are those elements of H, which lie entirely in $\bar{S}$. If II plays a strategy which prevents I from winning any of the $(\bar{S}, H(\bar{S}))$ for $S \in \mathcal{S}$, then it prevents I from winning the $A(1,1;\mathcal{P})$ also. Indeed, let us suppose that I succeeds in taking all elements (squares) of a translated copy of P_i for some $i = 1, ..., k$. If this copy of P_i has a common element with an $S \in \mathcal{S}$, then, from the diameter limitation, the whole copy lies within $\bar{S}$, i.e. I wins the game $(\bar{S}, H(\bar{S}))$, too.
One of the difficulties in establishing a strategy which guarantees a draw for II on every sub-board $\bar{S}$ is that the sub-boards are not disjoint. It means I's mark appears on nine of the sub-boards, and although II's answer is ninefold too, we cannot expect it to be the best on all of these sub-boards. We shall just ignore eight of them, and concentrating on one of them at a time, we create a relaxed $(25, 1; H(\bar{S}))$ game on every sub-board $\bar{S}$. Similarly to the idea of Lemma 3 of [10], we define a relation $\mathcal{O}$ on the set of the sub-boards, and use it to decide which sub-board should receive the mark of II. At the beginning of the game $\mathcal{O}$ is empty. We say $\bar{S}_u$ *owes* $\bar{S}_w$ if $(\bar{S}_u, \bar{S}_w) \in \mathcal{O}$. At the l^{th} step II selects a sub-board $\bar{S}^*$ such that:

1. $\bar{S}^*$ contains x_l, the last selection of I,
and
2. $\bar{S}^*$ does not owe any sub-board $\bar{S} \ni x_l$.

Then II updates the relation $\mathcal{O}$. $\bar{S}^*$ owes all $\bar{S} \neq \bar{S}^*$ which contain x_l, and non of these $\bar{S}$'s owe $\bar{S}^*$ in the updated relation. Now, if a sub-board $\bar{S} \ni x_l$ was not selected, then a (say) $\bar{S}^*$ was. $\bar{S}^*$ owes $\bar{S}$, and cannot be selected again until $\bar{S}$ is selected. Since at most 24 sub-boards may owe a sub-board $\bar{S}$, at least every 25^{th} step on every sub-board is answered.
Within a sub-board $\bar{S}$, selected by the previous rule, II plays accordingly to Lemma 1. It gives that II draws in *all* relaxed $(\bar{S}, H(\bar{S}))$ game, provided that

$$\sum_{A \in H(\bar{S})} 2^{-\frac{|A|}{25}} < \frac{1}{2}.$$

On the other hand $|H(\bar{S})| < 9d^2k$, so if

$$n \geq 50 \log_2 d + 25 \log_2 k + 100,$$

then the above inequality holds, therefore I cannot win. □

4 Conclusion

Upon proving Theorem 1, we have reached the limits of the weight function technique. On one hand, there is no reason to believe that winning sets of larger and larger diameter would really benefit I. On the other hand, the weight functions, unless an ingenious idea is incorporated, cannot help on the growing sub-boards. Indeed, Conjecture 1 is just a special case of an important open question in the theory of hypergraph games. As we mentioned earlier, in a number of cases the probabilistic heuristic works, that is one may prove a draw for II in the $(1,1,H)$-game, when a random argument shows the existence of a good 2-coloring of the hypergraph (X,H). It does not necessary break down when this existence of the good 2-coloring is guaranteed only by the *Lovász Local Lemma.* According to the Lovász Local Lemma there is a good 2-coloring of an even infinite hypergraph (X,H), if the maximum degree of (X,H) is "small" and the size of any $A \in H$ is "large" (see [6]). The natural direction of research is to find out if these conditions guarantee draw for the second player. Although there are very deep and promising results in [2] and [4] for the finite cases, the general solution is still far away.

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