Remarks on the Interval Number of Graphs

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Abstract

The interval number of a graph G is the least natural number t such that G is the intersection graph of sets, each of which is the union of at most t intervals. Here we propose a family of representations for the graph G, which yield the well-known upper bound $\lceil \frac{1}{2}(d+1) \rceil$, where d is the maximum degree of G. The extremal graphs for even d are also described, and the upper bound on the interval number in terms of the number of edges of G is improved.

1 Introduction and Results

It is a very natural idea to represent a graph G as the intersection graph of some sets. That is, we assign a set to each vertex of G so that v is adjacent to w if and only if the common part of the assigned sets is not empty. A $t-$*interval representation* is an assignment, where each set consists of at most t closed intervals. The *interval number* of G, denoted by $i(G)$, is the least integer t for which a t-representation of G exists. Finally, a representation is *displayed* if each set of the representation has an open interval disjoint from the other sets. Such an interval is called *displayed segment*.

There are a number of published results concerning bounds on $i(G)$, as well as applications of the interval representations [1-8]. Since for the complete graph K_n (on n vertices) $i(G) = 1$, the main interest lies in finding *upper bounds* in terms of the maximum degree, the number of vertices and the number of edges of a graph G, see in [2], [3], [6] and [8].

Theorem 1 (3) *If G is a graph with maximum degree d, then $i(G) \leq \lceil \frac{1}{2}(d+1) \rceil$.*

The bound of Theorem 1 is sharp, since the equality is attained for example a d-regular, triangle-free graphs G. We shall give a new proof of Theorem 1, which is also useful in investigating the extremal graphs of the degree bound.

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Theorem 2 *If a graph G has no d-regular, triangle-free component, then* $i(G) \leq \lceil \frac{1}{2} d \rceil$.

That is to say, in the case $d = 2k$ the extremal graphs are just the d-regular, triangle-free graphs. Unfortunately, one cannot expect to get such a simple result when $d = 2k+1$. For example the graph which arise from $K_{1,3}$ subdividing all its edges [7], or C_n, $n \geq 5$ with a chord have interval number 2 with $d = 3$.
It is possible to bound $i(G)$ in terms of e, where e is the number of edges in G. It was conjectured in [3] that $i(G) \leq \lceil \frac{1}{2}\sqrt{e}\,\rceil + 1$, which would be best possible because of the graphs $K_{2m,2m}$ for $m \in N$. The best published result is in [6], stating that $i(G) \leq \lceil \sqrt{\frac{e}{2}}\,\rceil + 1$. We shall improve on the estimations used in [6], and show

Theorem 3 *Every graph with e edges has a displayed interval representation with at most* $1 + \lceil \frac{2}{3}\sqrt{e}\,\rceil$ *intervals for each vertex.*

It is necessary to state one more earlier result in order to prove Theorem 3.

Theorem 4 (2) *If a graph G has $n > 1$ vertices, then* $i(G) \leq \lceil \frac{1}{4}(n+1) \rceil$*, and this bound is the best possible.*

2 Proofs

Proof of Theorem 1

We shall construct a displayed representation for the graph G such that at most $\lceil \frac{1}{2}(d(v)+1) \rceil$ intervals are assigned to each vertex v, where $d(v)$ designates the degree of the vertex v. A *walk* W in G is just a sequence of vertices $W = \{v_1, v_2, ..., v_l\}$ such that, there is an edge between v_i and v_{i+1} for each $i = 1, 2, ..., l-1$. Let us partition the edges of G into minimal number of edge disjoint walks $\{W_i\}_{i=1}^{j}$. Now represent the walk $W_i = (v_1^i, v_2^i, ..., v_{n(i)}^i)$ for $1 \leq i \leq j$, assigning an I_p^i interval to the vertex v_p^i such that two intervals have intersection if and only if the corresponding vertices are next to each other in the walk W_i. This procedure leads to a displayed interval representation of G. Since a vertex v can be an endvertex of the walks at most two times, if v is represented by l intervals, then $d(v) \geq 2(l-2) + 2 = 2l - 2$. Hence

$$\lceil \frac{1}{2}(d(v)+1) \rceil \geq \lceil \frac{1}{2}(2l-2+1) \rceil = \lceil l - \frac{1}{2} \rceil = l.$$

□

Proof of Theorem 2

We can assume that $d = 2k$ because of Theorem 1. Let us choose among all partitions of the edge set into a minimum number of edge disjoint walks a partition $\{W_i\}_{i=1}^{j}$ which also minimizes the size of the set Q of vertices occuring $k+1$ times in the walks $\{W_i\}_{i=1}^{j}$. The representation is the same as in the proof of Theorem 1.
If $Q = \emptyset$, we are done. For an $x \in Q$ there exists a $p \in \{1, ..., j\}$ such that $x = v_1^p$, $x = v_{n(p)}^p$ and $x \notin W_l$ for all $l \neq p$. The last statement follows from the minimality of j, since in case of $x = v_s^l \in W_l$ we could replace the walks W_p and W_l by the walk

$$W^* = (v_1^l, v_2^l, ..., v_s^l, v_2^p, ..., v_{n(p)}^p, v_{s+1}^l, ..., v_{n(l)}^l).$$

For any vertex $y = v_s^p \neq x$ from W_p, we can transform the walk W_p into the walk

$$W_p^* = (v_s^p, v_{s-1}^p, ..., v_1^p, v_{n(p)-1}^p, v_{n(p)-2}^p, ..., v_s^p).$$

That is, by the minimality of Q, y occurs in the walks $\{W_i\}_{i \neq p} \cup \{W_p^*\}$ $k+1$ times. Then again, all neighbors of y are in W_p. That is the vertex set of W_p is a $2k-$regular component of G. Now we can conclude the proof by showing that if a $2k-$regular graph G is not triangle-free, then $i(G) \leq k$. Suppose that u, v and w span a triangle in G. If $k = 1$, then $G = K_3$, and we are done. For $k > 1$ there is is an Euler circuit C in G, starting by v, u, w, v, x and finishing at v. But it can be represented by k intervals per vertex as in the proof of Theorem 1, just take the convex hull of the two intervals which represent v at the beginning of the walk.
□

Proof of Theorem 3

We need the definition of the *degree sequence* of a graph G first. Let us suppose that $v_1, ..., v_n$ is an order of the vertices of G such that $d_i \geq d_j$ if $i \leq j$, where $d_i = deg(v_i)$ denotes the degree of the vertex v_i. Our argument closely follows the one in [8]. The crucial difference is the additional information about the degree sequence of G. It is gained by using Theorem 1 and an idea, which first appeared in [4].

Lemma 1 *Let $d_1 \geq d_2 \geq ... \geq d_n$ be the degree sequence of a graph G. If $i(G) > t+1$, then $d_i \geq 2t - i + 1$.*

Proof of Lemma 1

Let v_i be a vertex of degree d_i. By Theorem 1

$$\lceil \frac{1}{2}(d_1 + 1) \rceil \geq i(G) > t + 1,$$

that is $d_1 \geq 2t + 2$. Now we partition the edges of G into directed forests, represent them one by one and remove the edges of the represented forest from G. The idea is that the representation of the l^{th} forest exhausts all edges adjacent to v_l, and decreases the degree of all vertices in the remaining graph which still has non zero degree. The construction of the first forest F_1 starts with choosing a breadth-first-search tree T_1, rooted in v_1, all edges directed toward v_1. If there are vertices outside of T_1, just pick arbitrary trees in which the edges are directed toward the root. The procedure for selecting F_l is similar, we take v_l, the vertex of degree d_l as a root of a tree, and also take other trees if the remaining graph is not connected. The main point is that F_l is maximal, and all edges adjacent to v_l are in $F_1 \cup ... \cup F_l$. For the maximum degree Δ^i in the remaining graph $G^i = G \backslash F_1 \cup ... \cup F_{i-1}$ we have show that

$$\Delta^i \leq d_i - (i-1)$$

by induction. On the other hand, we can represent the edges of $F_1 \cup ... \cup F_l$ by using at most $l+1$ intervals for each vertex. First assign intervals I_v to each vertex v of G such that $I_v \cap I_w = \emptyset$ for $v \neq w$. Then, for each $i \in \{1, ..., l\}$ if the directed edge (v, w) is in F_i, assign a small interval to v inside in I_w, which has no common points with the other intervals.
Because of Theorem 1 and the previous representation we have

$$i(G) \leq i + i(G \backslash F_1 \cup ... \cup F_{i-1}) \leq i + \lceil \frac{d_i - (i-1) + 1}{2} \rceil.$$

Since $t + 1 < i(G) \leq i + \lceil \frac{d_i - i + 2}{2} \rceil$, it follows that

$$t + 3/2 \leq i + \frac{d_i - i + 2}{2},$$

that is $d_i \geq 2t - i + 1$.
□

Now, with a few modifications, we may repeat the argument presented in [8]. First, partition the vertices of G into two classes, A and B. A contains the vertices of degree at least $\lceil \frac{2}{3}\sqrt{e} \rceil + 1$, while the degree of a vertex from B is at most $\lceil \frac{2}{3}\sqrt{e} \rceil$.
The edges between the elements of A can be represented by using at most $\lceil \frac{1}{4}(|A|+1) \rceil$ intervals for each vertex because of Theorem 4. Let us make this system of intervals displayed by adding an isolated interval for each vertex of G in a same way as in the proof of Lemma 1. For each edges between A and B, or inside B, take an endpoint from B, and place a small interval for it into a displayed segment for its neighbor. This procedure produces at most $\lceil \frac{2}{3}\sqrt{e} \rceil + 1$ intervals for an element of B. That is

$$i(G) \leq \max(\lceil \frac{2}{3}\sqrt{e} \rceil + 1, \lceil \frac{1}{4}(|A| + 1) \rceil + 1).$$

In order to estimate $|A| = k$, we need the identity $2e = \sum_{i=1}^{n} d_i$, where $\{d_i\}_{i=1}^{n}$ is the degree sequence in decreasing order. There is nothing to prove if $i(G) \leq \lceil \frac{2}{3}\sqrt{e} \rceil + 1$, so we may assume that

$$d_i \geq 2\lceil \frac{2}{3}\sqrt{e} \rceil - i + 1$$

by Lemma 1. Thus

$$2e = \sum_{i=1}^{\lceil \frac{2}{3}\sqrt{e}\,\rceil} d_i + \sum_{i=\lceil \frac{2}{3}\sqrt{e}\,\rceil+1}^{k} d_i + \sum_{i=k+1}^{n} d_i,$$

which implies

$$2e \geq \sum_{i=1}^{\lceil \frac{2}{3}\sqrt{e}\,\rceil} (2\lceil \frac{2}{3}\sqrt{e}\,\rceil - i + 1) + \sum_{i=\lceil \frac{2}{3}\sqrt{e}\,\rceil+1}^{k} (\lceil \frac{2}{3}\sqrt{e}\,\rceil + 1).$$

Simple computation shows that $k \leq \frac{8}{3}\sqrt{e} - 1$. Plugging in this estimation, one gets the bound

$$i(G) \leq \max(\lceil \frac{2}{3}\sqrt{e}\rceil + 1, \lceil \frac{1}{4}(\frac{8}{3}\sqrt{e} - 1 + 1)\rceil + 1) \leq \lceil \frac{2}{3}\sqrt{e}\,\rceil + 1.$$

□

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