A Sharp Edge Bound on the Interval Number of a Graph *

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Abstract

The interval number of a graph G, denoted by $i(G)$, is the least natural number t such that G is the intersection graph of sets, each of which is the union of at most t intervals. Here we settle a conjecture of Griggs and West about bounding $i(G)$ in terms of e, that is the number of edges in G. Namely, it will be shown that $i(G) \leq \lceil \sqrt{e}/2 \rceil + 1$. It is also observed that the edge bound induces $i(G) \leq \sqrt{3\gamma(G)/2} + O(1)$, where $\gamma(G)$ is the genus of G.

1 Introduction and Results

In this paper we are concerned with representing graphs as special intersection graphs. That is, we assign a set to each vertex of G so that v is adjacent to w if and only if the assigned sets intersect. A *t-interval representation* is such an assignment in which each set consists of at most t closed intervals. The *interval number* of G, denoted by $i(G)$, is the least integer t for which a t-representation of G exists. We need one more definition: a representation is *displayed* if each set of the representation contains an open interval disjoint from the other sets. Such an interval is a *displayed segment.*

Several interesting aspects of the interval number have been investigated before; here we pursue only one of these. More than one and a half decade ago Griggs and West conjectured that $i(G) \leq \lfloor \sqrt{e}/2 \rfloor + 1$ for any graph G, where e is the number of edges in G. They also showed that this would be the best possible, because the bound is attained for the complete bipartite graphs $K_{2m,2m}$, where m is a positive integer. In fact we restate their conjecture as $i(G) \leq \lceil \sqrt{e}/2 \rceil + 1$, since its value at most one bigger than of the original, while two forms coincide for the previous graphs. In a series of papers it was proved that $i(G) \leq \lceil c\sqrt{e} \rceil + 1$ for $c = 1$, $c = \sqrt{2}/2$ and $c = 2/3$, see [6], [11] and [8]. The main goal of this paper is to conclude this process by showing that the right constant is really $c = 1/2$. So if one looks for a bound on $i(G)$ in form of $\lceil c_1 \sqrt{e(G)} \rceil + c_2$, then the previous bound is sharp. (In [11] an other related conjecture is mentioned, namely that $i(G) \leq \lceil (\sqrt{e} + \sqrt{e^{-1}})/2 \rceil$, which can be a little better than the version of Theorem 1 when e is not a square of an even integer.)

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We also mention some related problems that are worth further study.

Theorem 1 *Every graph with e edges has a displayed interval representation with at most $\lceil \sqrt{e}/2 \rceil + 1$ intervals for each vertex.*

There is a relationship between the interval number of a graph G and $\gamma(G)$, the *genus* of G. It was shown in [9] that $i(G) \leq \left\lceil \sqrt{3\gamma(G)} \right\rceil + 3$. As a consequence of Theorem 1 this result can be improved:

Theorem 2 *For a graph G, $i(G) \leq \left\lceil \sqrt{3\gamma(G)/2} \right\rceil + 5$, provided the genus is big enough.*

2 Important Lemmas

The proof of Theorem 1 is based on two previous results, which bound $i(G)$ in terms of the arboricity and the number of vertices of G, see [2], [5] and [9]. However, we need improvements on the bound in term of the arboricity, partly in order to "smooth out" the subsequent calculations. The arboricity of a graph G, denoted by $\Upsilon(G)$, is the smallest number of classes needed partitioning the edge set of G into forests. We shall use $e(G)$ and $v(G)$ to designate the number of edges and vertices of a graph G, respectively. The following lemma is due to Tom Trotter, although it appeared in [9].

Lemma 1 *[9] A graph G has a displayed interval representation with at most $\Upsilon(G)+1$ intervals for each vertex.*

The proof of Lemma 1 yields a technically stronger statement.

Lemma 2 *Beginning with pairwise disjoint intervals assigned to the vertices of a graph G, there exists a displayed $\Upsilon(G) + 1$-representation in which a specified vertex v is assigned no additional intervals.*

Proof. Given a decomposition of G into $\Upsilon(G)$ forests, we orient the edges of each forest toward v (or toward an arbitrary fixed vertex in each tree not containing v) and represent each edge by placing a small interval for the tail within the displayed portion of the original interval for the head. $\square$

Another variation of Lemma 1 replaces the arboricity with the maximum edge density of G. Although its proof is very similar to Lemma 1 we shall sketch it for the sake of completeness. Here $\rho(G)$ denotes $\max_{\emptyset \neq H \subset G} e(H)/v(H)$, the maximum density of the graph G.

Lemma 3 *A graph G has a displayed interval representation with at most $\lceil \rho(G) \rceil + 1$ intervals for each vertex.*

Proof. Let $k = \max_{\emptyset \neq H \subset G} \lceil e(H)/v(H) \rceil$. Using the method of Lemma 2, it suffices to prove that G decomposes into k subgraphs with at most one circuit and such a graph can be oriented with each vertex having outdegree at most one.

To obtain the decomposition, note that the subsets of $E(G)$ containing at most one circuit of G are independent sets of a matroid M. Because such a subgraph is formed by adding one edge to an acyclic subgraph, M is the Nash-Williams sum of the circuit matroid of G and the uniform matroid $U_{e,1}$, where $e = e(G)$. According to the formulas of Nash-Williams or Edmonds, a matroid $M = (S, \mathcal{I})$ can be partitioned into k independent sets iff $k \geq \max_{\emptyset \neq X \subset S} \lceil |X|/rank(X) \rceil$, see [7] and [4]. In our case this implies that $E(G) = E_1 \cup ... \cup E_k$, where for $1 \leq i \leq k$ the subset E_i contains at most one circuit.

To verify the orientation property, observe that components not containing circuits can be directed toward an arbitrary vertex. If a component has a circuit containing an edge uv, orient this edge toward u and all edges of the tree obtained deleting uv toward v. $\square$

Remark. The existence of an orientation of G having maximum outdegree at most $d = \lceil \rho(G) \rceil$ might have been discovered several times before. In [1] Alon and Tarsi, expressing the same opinion, also give a proof of it via the celebrated König-Hall theorem.

Another fundamental result what we need is a bound on the interval number in terms of the number of vertices.

Theorem 3 *[5] If a graph G has $n > 1$ vertices, then $i(G) \leq \lceil (n+1)/4 \rceil$, and this bound is the best possible.*

A stronger version of Theorem 3 was proved in [2]. Alas, the representation used in the proof of Theorem 3 is not displayed. It is possible to prove an extension of it, since, as it was first noted in [11], the induction part of the proof preserves the displayed feature. Thus the proof of Theorem 3 in [5] actually yields a stronger statement, namely:

Theorem 4 *If a graph G has $n > 3$ vertices, then G has a displayed interval representation with at most $i(G) \leq \lceil (n+1)/4 \rceil$ intervals for each vertex.*

3 The Main Result

The representation of the desired type is made up in two steps. First a dense subgraph of G is represented by means of Theorem 4, then an appropriate extension is given by the line of Lemma 2. We need some notation and a new lemma in order to formalize this. If Q is an induced subgraph of G, then G/Q stands for the contraction of Q in G such that the loops are deleted. Note that Q is not necessarily connected and G/Q may have multiple edges. In obtaining $E(G/Q)$ from $E(G \setminus Q)$, for each $u \notin V(Q)$

we add (u,q) with multiplicity equal to the number of neighbors of u in $V(Q)$, where q designates the "contraction" of Q. More specifically $G/Q = (V(G/Q), E(G/Q))$, where

$$V(G/Q) = \{V(G) \setminus V(Q)\} \cup \{q\}$$

and

$$E(G/Q) = E(G \setminus Q) \cup \{(u,q): \ (u,v) \in E(G), \ u \notin V(Q), \ v \in V(Q)\}.$$

It is very convenient to use the notion of $\hat{\imath}(G)$, the displayed interval number of G, which is defined like $i(G)$ except that the representations must be displayed, see in [2].

Lemma 4 *Let Q be an induced subgraph of G. Then $\hat{\imath}(G) \leq \max\{\hat{\imath}(Q), \Upsilon(G/Q)+1\}$.*

Proof. Let us take a displayed $\hat{\imath}(G)$-representation f of Q, then assign pairwise disjoint intervals to each $u \in V(G) \setminus V(Q)$ that intersect no intervals used in f. By Lemma 2, the edges of G/Q can be represented such that at most $\Upsilon(G/Q)$ new intervals are used for each vertex $u \in V(G) \setminus V(Q)$, and no new intervals are used for q. Consequently the set $E(G) \setminus E(Q)$ can also be represented this way, since the representation f is displayed. (For each $(u,v) \in E(G)$ such that $v \in V(Q)$ and $u \in V(G) \setminus V(Q)$, one can put a small interval assigned to u into the displayed segment of the interval corresponding to v in f.) $\square$

Proof of Theorem 1

Let us set $k = \lceil \sqrt{e(G)}/2 \rceil$, and let Q be a maximal subgraph of G such that $e(Q)/v(Q) \geq k - 1/2$. If $Q = \emptyset$, i.e. the maximum edge density of G is less than k, then Lemma 3 implies Theorem 1, so we may assume that $Q \neq \emptyset$. Because of the maximality of Q, $e(H)/v(H) < k - 1/2$ for every $H \subset G/Q$.

We also claim that $e(H)/(v(H)-1) \leq k$ for $H \subset G/Q$. Let q be the vertex of G/Q representing Q; if $q \in V(H)$, then $e(H)/(v(H)-1) > k$ would contradict the maximality of Q. If $q \notin H$, then $e(H)/(v(H)-1) > k$ would imply $v(H) > 2k$, since $(v(H)-1)v(H)/2 \geq e(H)$. With this we have

$$\frac{e(H)}{v(H)-1} = \frac{e(H)}{v(H)} \frac{v(H)}{v(H)-1} < \left(k - \frac{1}{2}\right)\left(1 + \frac{1}{2k}\right) < k.$$

We have proved that $\Upsilon(G/Q) \leq k = \lceil \sqrt{e(G)}/2 \rceil$. Finally, we need to bound $\hat{\imath}$(Q) in order to conclude the proof by Lemma 4. This is done by Theorem 4, since we may assume that $v(Q) > 3$.

$$\hat{\imath}(Q) \leq \left\lceil \frac{1}{4}(v(Q)+1) \right\rceil \leq \left\lceil \frac{1}{4}(\frac{e(Q)}{k-1/2}+1) \right\rceil \leq$$

$$\leq \left\lceil \frac{1}{2}\sqrt{e(G)} + \frac{1}{2\sqrt{e(G)}-2} + \frac{1}{2} \right\rceil \leq \left\lceil \frac{1}{2}\sqrt{e(G)} \right\rceil + 1,$$

provided that $e(G) > 9$.

When $e(G) \le 9$, one can give an explicit displayed $(\lceil\sqrt{e(G)}/2\rceil+1)$-representations for Q, if $e(G) \le 9$. $\square$

4 An Application and Remarks

There are several ways to get Theorem 2 by the application of the tools developed in this paper. Here we reduce it to Theorem 1.

Proof of Theorem 2. We may assume that all vertices of G has degree at least $\sqrt{3\gamma(G)/2}$. (An analogous idea appears in [11], since in displayed representations all edges incident to a low-degree vertex v can be represented by intervals all assigned to v.) To simplify notation, we use n, e and γ instead of $v(G)$, $e(G)$ and $\gamma(G)$ in what follows. Our minimum-degree assumption implies that $n\sqrt{3\gamma/2} \le 2e$. On the other hand, Euler's formula implies

$$3n + 6\gamma - 6 \ge e.$$

Plugging the bound on n into this yields

$$\frac{6e}{\sqrt{3\gamma/2}} + 6\gamma - 6 \ge e,$$

which gives

$$e \le \frac{6\gamma - 6}{1 - 6/\sqrt{3\gamma/2}} \le 6\gamma + 30\sqrt{\gamma},$$

if γ is big enough. By Theorem 1

$$i(G) \le \left\lceil \frac{1}{2}\sqrt{e} \right\rceil + 1 \le \left\lceil \frac{1}{2}\sqrt{6\gamma + 30\sqrt{\gamma}} \right\rceil + 1 \le \left\lceil \sqrt{3\gamma/2} \right\rceil + 5,$$

and we are done. $\square$

Since $\gamma(K_n) = \lceil (n-3)(n-4)/12 \rceil$, one cannot improve this result by using Theorem 1 alone. However, we believe that the right constant is one.

Conjecture 1 *Every graph with genus γ has a displayed interval representation with at most $\lceil \sqrt{\gamma}\, \rceil + 3$ intervals for each vertex.*

One may get closer to this by taking into account that the interval number is *not* a monotone function. We can formulate another conjecture along these lines which looks interesting in its own right. Let us recall that $\overline{G}$ is the complement of a graph G and $e(\overline{G})$ is the number of edges in $\overline{G}$.

Conjecture 2 *Every graph G has a displayed interval representation with at most $\left\lceil \sqrt{e(\overline{G})/2} \right\rceil + 1$ intervals for each vertex.*

In fact Conjecture 2 and Lemma 3 would imply Theorem 1. Furthermore Theorem 1 and Conjecture 2 immediately give almost the same bound as Theorem 3, which makes our claim very desirable, and probably hard. A step towards Conjecture 2 was made in [3] showing that $i(G) \leq \sqrt{e(\overline{G})/2} + o(n)$.

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