On Chooser-Picker positional games¹

András Csernenszky^a,

C. Ivett Mándity^a

András Pluhár^{a,*}

^aDepartment of Computer Science, University of Szeged, 6720 Szeged, Hungary

,

Abstract

Two new versions of the so-called Maker-Breaker Positional Games are defined by József Beck. In these variants Picker takes unselected pair of elements and Chooser keeps one of these elements and gives back the other to Picker. In the Picker-Chooser version Picker is Maker and Chooser is Breaker, while the roles are swapped in the Chooser-Picker version. It seems that both the Picker-Chooser and Chooser-Picker versions are not worse for Picker than the original Maker-Breaker versions. Here we give winning conditions for Picker in some Chooser-Picker games that extend the results of Beck.

Key words: Hypergraph, positional game, Picker-Chooser game, Beck's conjecture

1 Introduction

Recall that formally a Positional Game, or a *Maker-Maker hypergraph game* is defined as follows. Given an arbitrary hypergraph (V, \mathcal{F}) , the first and second players take elements of V in turns. The player, who takes all the elements of an edge $A \in \mathcal{F}$ first wins the game. It is well known that, assuming perfect play, either the first player wins or the game is a draw. The theory of Positional Games is quite well developed, here we can recall only some results. For further

Preprint submitted to Discrete Mathematics

^{*} Corresponding author.

Email addresses: csernenszkya@gmail.com (András Csernenszky),

mivett@inf.u-szeged.hu (C. Ivett Mándity), pluhar@inf.u-szeged.hu (András Pluhár).

¹ The research was partially supported by OTKA grants T034475 and T049398.

readings see the important works of Berlekamp, Conway and Guy [9] or Beck [7].

The Maker-Breaker version of a Positional Game on a hypergraph (V, \mathcal{F}) is as follows. The players take the elements of V as before, but Maker wins by taking all the elements of an $A \in \mathcal{F}$, while Breaker wins otherwise. This approach has proved to be quite useful, since if Breaker wins (as a second player) then the original game is a draw, while if the first player wins the original game then Maker wins the Maker-Breaker version. More examples are in [3–5,12,13].

An important guide to understanding Maker-Breaker games is the so-called *probabilistic intuition*, for more details and examples by Beck, Bednarska et al, Krivelevich [5,8,14]. Roughly speaking, we distribute the elements of V among Maker and Breaker randomly, and expect the win of that player in the original Maker-Breaker game whose winning chance is greater in the random play. This simple heuristic works surprisingly often, and gives useful instructions even when it fails (see [2]).

Studying the very hard clique games, Beck [6] introduced a different type of heuristic, that proved to be a great success. He defined the *Picker-Chooser* or shortly P-C and the *Chooser-Picker* (C-P) versions of a Maker-Breaker game that resembles fair division, (see [20]). In these versions Picker takes an unselected pair of elements and Chooser keeps one of these elements and gives back the other to Picker. In the Picker-Chooser version Picker is Maker and Chooser is Breaker, while the roles are swapped in the Chooser-Picker version. When |V| is odd, the last element goes to Chooser. Beck obtained that conditions for winning a Maker-Breaker game by Maker and winning the Picker-Chooser version of that game by Picker coincide in several cases. Furthermore, Breaker's win in the Maker-Breaker and Picker's win in the Chooser-Picker version seem to occur together.

Beck [6] has another interesting remark, namely that Picker may win easily the Picker-Chooser game if Maker wins the corresponding Maker-Breaker game. He formulates this as follows:

"Note that Picker has much more control in the Picker-Chooser version than Chooser does in the Chooser-Picker version, or Maker does in the Maker-Breaker version so the Picker-Chooser game is far the simplest case. This relative simplicity explains why we start with the Picker-Chooser game instead of the perhaps more interesting Maker-Breaker game."

However, one has to be careful to spell out a good conjecture, since it is easy to check that Chooser wins the 2×2 hex.

The precise form of Beck's conjecture is:

Conjecture 1 Picker wins a Picker-Chooser (Chooser-Picker) game on (V, \mathcal{F}) if Maker (Breaker) as second player wins the corresponding Maker-Breaker game.

Remarks. It is enough to prove Conjecture 1 for Picker-Chooser games since the Chooser-Picker case would follow. To see this one just considers (V, \mathcal{F}^*) , the *transversal hypergraph* of (V, \mathcal{F}) . That is \mathcal{F}^* contains those minimal sets $B \subset V$ such that for all $A \in \mathcal{F}$, $A \cap B \neq \emptyset$. Note that Breaker as a first (second) player wins the Maker-Breaker (V, \mathcal{F}) iff Maker as a first (second) player wins the Maker-Breaker (V, \mathcal{F}^*) .

The decision problem that if Picker wins a P-C (or C-P) game is at least NP-hard [10], but probably it is PSPACE-complete as that of the Maker-Breaker games, shown by Schaefer [18]. Still, for concrete games it can be easier to decide the outcome of the P-C (C-P) version than the Maker-Maker version. That is if Conjecture 1 is proved for a class of hypergraphs then the easier P-C (C-P) games can be used in an alpha-beta pruning algorithm for the harder Maker-Breaker game. A natural class for that is the otherwise hopeless Hales-Jewett or torus games for low dimension (see [7,13]). We discuss some examples and useful tools for that direction in Section 2. Here we would emphasize the extension of Picker-Chooser games to infinite hypergraphs and the role of Lemma 8 and Proposition 9 in this case. These might be used in solving Harary-type of polyomino problems for Chooser-Picker games for which the Maker-Breaker versions were studied by Harary, Blass, Pluhár and Sieben [9,17,19].

Then we prove Conjecture 1 for the Picker-Chooser version of Shannon switching game in the generalized version as Lehman did in [15]. Let (V, \mathcal{F}) be a matroid, where \mathcal{F} is the set of bases, and Picker wins by taking an $A \in \mathcal{F}$. Note that this is equivalent with the Chooser-Picker game on (V, \mathcal{C}) , where \mathcal{C} is the collection of *cutsets* of the matroid (V, \mathcal{F}) , that is for all $A \in \mathcal{F}$ and $B \in \mathcal{C}, A \cap B \neq \emptyset$.

Theorem 2 Let \mathcal{F} be collection of bases of a matroid on V. Picker wins the Picker-Chooser (V, \mathcal{F}) game, if and only if there are $A, B \in \mathcal{F}$ such that $A \cap B = \emptyset$.

The Erdős-Selfridge theorem [11] gives a very useful condition for Breaker's win in a Maker-Breaker (V, \mathcal{F}) game.

Theorem 3 (Erdős-Selfridge [11]) Breaker as the second player has a winning strategy in the Maker-Breaker (V, \mathcal{F}) game when

$$\sum_{A\in\mathcal{F}} 2^{-|A|} < \frac{1}{2}.$$

Using a stronger condition, Beck [6] proves Picker's win in a Chooser-Picker (V, \mathcal{F}) game. (For the P-C version he proved a sharp result that we include here.) Let $||\mathcal{F}|| = \max_{A \in \mathcal{F}} |A|$ be the rank of the hypergraph (V, \mathcal{F}) .

Theorem 4 [6] If

$$T(\mathcal{F}) := \sum_{A \in \mathcal{F}} 2^{-|A|} < \frac{1}{8(||\mathcal{F}||+1)},\tag{1}$$

then Picker has an explicit winning strategy in the Chooser-Picker game on hypergraph (V, \mathcal{F}) . If $T(\mathcal{F}) < 1$, then Chooser wins the Picker-Chooser game on (V, \mathcal{F}) .

We improve on his result by showing:

Theorem 5 If

$$\sum_{A \in \mathcal{F}} 2^{-|A|} < \frac{1}{3\sqrt{||\mathcal{F}|| + \frac{1}{2}}},\tag{2}$$

then Picker has an explicit winning strategy in the Chooser-Picker game on hypergraph (V, \mathcal{F}) .

It is worthwhile to spell out a special case of Conjecture 1 for this case, that would be the sharp extension of Erdős-Selfridge theorem to Chooser-Picker games.

Conjecture 6 If

$$\sum_{A\in\mathcal{F}} 2^{-|A|} < \frac{1}{2},$$

then Picker wins the Chooser-Picker game on the hypergraph (V, \mathcal{F}) .

The rest of the paper is organized as follows. We extend Conjecture 1 to infinite games, and discuss the classical k-in-a-row games in Section 2. Finally, in Sections 3 and 4 we prove Theorem 2 and Theorem 5, respectively.

2 The k-in-a-row

The game k-in-a-row is played on the infinite square grid ("graph paper"), and the players' goal is to get k squares in a row vertically, horizontally or diagonally first. By a strategy stealing argument the first player wins or achieves a draw for all $k \in \mathbb{N}$. Moreover the first player wins if $k \leq 4$, and the game is a blocking draw if $k \geq 8$, see e. g. in [7,9,12]. A delicate case study by Allis [1] shows that the first player wins for k = 5 on the 15×15 board. While the k = 5 is still open on the infinite board, Allis' result implies that Maker wins for k = 5 in the Maker-Breaker version.

The Picker-Chooser k-in-a-row is an easy Picker's win for all $k \in \mathbb{N}$, by Beck's argument in [7]. The Chooser-Picker is again Picker's win for k > 1 on the infinite board, since Picker may select elements far from each other at all time. However, the games become interesting if we restrict them to a finite board, since sooner or later all elements must be selected. (One might think that Chooser starts the game by selecting a finite part of the board.)

Proposition 7 Picker wins the Chooser-Picker version of the game 8-in-arow on any $B \subseteq \mathbb{Z}^2$.

Proof. First we need an easy but useful lemma. Given the hypergraph (V, \mathcal{F}) let $(V \setminus X, \mathcal{F}(X))$ denote the hypergraph where $\mathcal{F}(X) = \{A \in \mathcal{F}, A \cap X = \emptyset\}$.

Lemma 8 If Picker wins the Chooser-Picker game on (V, \mathcal{F}) , then Picker also wins it on $(V \setminus X, \mathcal{F}(X))$.

Proof. By induction it is enough to prove the statement for $X = \{x\}$, i. e., |X| = 1. Assume that p is a winning strategy for Picker in the game on (V, \mathcal{F}) . That is, in a certain position of the game, the value of the function p is a pair of unselected elements that Picker is to give to Chooser. We can modify p in order to get a winning strategy p^* for the Chooser-Picker game on $(V \setminus \{x\}, \mathcal{F}(\{x\}))$.

Let us follow p while it does not give a pair $\{x, y\}$. Getting a pair $\{x, y\}$, we ignore it, and pretend we are playing the game on (V, \mathcal{F}) , where Chooser has taken y and has returned x to us. If |V| is odd, there is a $z \in V$ at the end of the game that would go to Chooser. Here Picker's last move is the pair $\{y, z\}$. Picker wins, since Chooser could not win from this position even getting the whole pair $\{y, z\}$. If |V| is even, p^* leads to a position in which y is the last element, and it goes to Chooser. But the outcome is then the same as the outcome of the game on (V, \mathcal{F}) , that is Picker's win.

We shall cut up the infinite board to sub-boards in the same way as was in [12], see also Figure 1. The left tile and its mirror image are the bases of the tiling. The winning sets for the these sub-boards are the rows, the diagonals of slope one, and the two pairs indicated by the thin lines. The middle of the picture shows the tiling itself. We use one type of tile in an infinite strip, and its mirror image in the neighboring stripes. On the right side of Figure 1 the transformed tile is drawn, where the winning sets are the rows, columns and the indicated two pairs.



Figure 1. The subdivison of the plane.

Let \overline{B} be the union of those sub-boards meeting B. We show that Picker wins the Chooser-Picker 8-in-a-row game for the board \overline{B} . Note that \overline{B} is a union of sub-boards. Picker plays auxiliary games on the sub-boards independently of each other with the goal of preventing Chooser from getting a winning set of a sub-board.

To achieve this goal, Picker selects the two pairs first on any sub-board, that give rise to the possible positions shown in Figure 2. Then Picker uses the appropriate winning pairing strategy indicated by the thin lines. One checks easily that if Picker wins all the auxiliary games then he wins the Chooser-Picker 8-in-a-row game on playing \bar{B} , too. Finally, by Lemma 8, Picker wins on B.



Figure 2. Pairings on a sub-board.

One might wonder how the idea of the pairings used in Proposition 7 came from. It is worthwhile to spell out the following simple fact.

Proposition 9 In a Chooser-Picker game if a winning set contains no elements of Picker, and has only two untaken elements, x, y then Picker has an optimal strategy that starts with picking the pair $\{x, y\}$.

Proof. We may assume that Picker has a winning strategy p, otherwise there is nothing to prove. First we show that during any optimal play of the game Picker has to offer the pair $\{x, y\}$ sometimes. If Picker offers, say, $\{x, z\}, z \neq y$, and y has not been taken yet, Chooser would keep x, and win taking y later. Now let us assume the Chooser has a winning strategy ρ , taking, say, x if Picker starts with $\{x, y\}$. Chooser can adapt the strategy ρ against any strategy of Picker's by pretending that the start was $\{x, y\}$. Over the course of the play Picker has to offer the pair $\{x, y\}$. Then Chooser takes x and resumes playing the strategy ρ , and Chooser wins, since the outcome of the game would be the same if Picker would have started with $\{x, y\}$.

3 Proof of Theorem 2

The notation and the proof closely follow the ones given in [16] for the Maker-Breaker case.

First we show that if there are no two disjoint $A, B \in \mathcal{F}$ then Chooser wins. Let $\mathcal{M}_1 = (V, \mathcal{F})$ and $\mathcal{M} = \mathcal{M}_1 \vee \mathcal{M}_1$ be the union matroid of \mathcal{M}_1 with itself. The rank function $r_{\mathcal{M}}$ of the union matroid of $\mathcal{M} = M_1 \vee \cdots \vee M_k$ is the following,

$$r_{\mathcal{M}}(S) = \min_{T \subset S} \left\{ |S \setminus T| + \sum_{i=1}^{k} r_i(T) \right\},\$$

where the matroids are defined on the same ground set S, and the matroid \mathcal{M}_i has the rank function r_i . We have $\min_{T \subset V} \{|V \setminus T| + 2r_1(T)\} = r_{\mathcal{M}}(V) < 2r_1(V)$, since \mathcal{M}_1 does not have two disjoint bases. Equivalently, $|V \setminus T| < 2(r_1(V) - r_1(T))$. Receiving a pair (x, y), Chooser keeps an element of $V \setminus T$ if possible. At the end of the game Chooser owns at least $\lceil |V \setminus T|/2 \rceil$ elements of $V \setminus T$. That is Picker may own at most $\lfloor |V \setminus T|/2 \rfloor < r_1(V) - r_1(T)$ elements of $V \setminus T$ at the end of the game.

Let Y be the elements of Picker at the end of the game. Clearly,

$$r_1(Y) \le r_1(Y \cap (V \setminus T)) + r_1(T) < r_1(V) - r_1(T) + r_1(T) = r_1(V),$$

that is Picker has lost the game.

For the other direction, we assume that $A, B \in \mathcal{F}, A \cap B = \emptyset$, and use induction. We consider the matroid $\mathcal{M}/y \setminus x$ given a pair (x, y) taken by Chooser and Picker, respectively. Clearly Picker wins the game for \mathcal{M} if he can win it for $\mathcal{M}/y \setminus x$. (The dimension of $\mathcal{M}/y \setminus x$ is one less than that of \mathcal{M} , and if A' is a base of $\mathcal{M}/y \setminus x$, then $A' \cup \{y\}$ is a base of \mathcal{M} .)

All we need here is the strong base exchange axiom (or rather theorem), that says if A and B are bases of a matroid \mathcal{M} , then there exist $x \in A, y \in B$ such that both $\{A \setminus \{x\}\} \cup \{y\}$ and $\{B \setminus \{y\}\} \cup \{x\}$ are also bases of \mathcal{M} . Picker selects the pair (x, y) such that the above applies, and reduces the game to either $\mathcal{M}/y \setminus x$ or $\mathcal{M}/x \setminus y$. Since $A \setminus \{x\}$ and $B \setminus \{y\}$ are disjoint bases both in $\mathcal{M}/y \setminus x$ and $\mathcal{M}/x \setminus y$, we can proceed. \Box

4 Proof of Theorem 5

We shall modify the proof of Theorem 4 appropriately. The idea of the proof is to associate a weight function $T(\mathcal{F})$ to a hypergraph (V, \mathcal{F}) that measures the danger for Picker. The value of T becomes 1 iff Chooser wins the game, so Picker tries to keep T down. In Maker-Breaker games the greedy selection works, see the classical Erdős-Selfridge theorem in [11] or in [7]. Let $T(\mathcal{F}) =$ $\sum_{A \in \mathcal{F}} 2^{-|A|}, T(\mathcal{F}; v) = \sum_{v \in A \in \mathcal{F}} 2^{-|A|}$ and $T(\mathcal{F}; v, w) = \sum_{\{v,w\} \subset A \in \mathcal{F}} 2^{-|A|}$ for an arbitrary hypergraph (V, \mathcal{F}) .

Assume that after the *i*th turn Chooser already has the elements x_1, x_2, \ldots, x_i and Picker has the elements y_1, y_2, \ldots, y_i . Now Picker picks a 2-element set $\{v, w\}$, from which Chooser will choose x_{i+1} , and the other one (i. e. y_{i+1}) will go back to Picker. Let $X_i = \{x_1, x_2, \ldots, x_i\}$ and $Y_i = \{y_1, y_2, \ldots, y_i\}$. Let $V_i = V \setminus (X_i \cup Y_i)$. Clearly $|V_i| = |V| - 2i$. Let $\mathcal{F}(i)$ be the truncated subfamily of \mathcal{F} which consists of the unoccupied parts of the still dangerous winning sets:

$$\mathcal{F}(i) = \{A \setminus X_i : A \in \mathcal{F}, |A \setminus X_i| \le \lceil |V_i|/2 \rceil, A \cap Y_i = \emptyset\}.$$

Here we will deviate a little from Beck's proof, since he includes all sets $A \in \mathcal{F}$, $|A \setminus X_i| \leq |V_i|$ in $\mathcal{F}(i)$ if $A \cap Y_i = \emptyset$. But if $|A \setminus X_i| > \lceil |V_i|/2 \rceil$, then Picker *automatically* gets an element of A, so deleting these sets from $\mathcal{F}(i)$ does not change the outcome of the game.

Let $\mathcal{F}(end) = \mathcal{F}(\lceil |V|/2 \rceil)$, i. e., these are the unoccupied parts of the still dangerous sets at the end of the play. Chooser wins iff $T(\mathcal{F}(end)) \ge 1$, so to guarantee Picker's win it is enough to show that $T(\mathcal{F}(end)) < 1$. Let x_{i+1} and y_{i+1} denote the (i + 1)th elements of Chooser and Picker, respectively. Then we have

$$T(\mathcal{F}(i+1)) = T(\mathcal{F}(i)) + T(\mathcal{F}(i); x_{i+1}) - T(\mathcal{F}(i); y_{i+1}) - T(\mathcal{F}(i); x_{i+1}, y_{i+1}).$$

It follows that

$$T(\mathcal{F}(i+1)) \le T(\mathcal{F}(i)) + |T(\mathcal{F}(i); x_{i+1}) - T(\mathcal{F}(i); y_{i+1})|.$$

Introduce the function

$$g(v,w) = g(w,v) = |T(\mathcal{F}(i);v) - T(\mathcal{F}(i);w)|$$

which is defined for any 2-element subset $\{v, w\}$ of V_i . Picker's next move is that 2-element subset $\{v_0, w_0\}$ of V_i for which the function g(v, w) achieves its minimum. Since $\{v_0, w_0\} = \{x_{i+1}, y_{i+1}\}$, we have

$$T(\mathcal{F}(i+1)) \le T(\mathcal{F}(i)) + g(i), \tag{3}$$

where

$$g(i) = \min_{v, w: v \neq w, v, w \subset V_i} |T(\mathcal{F}(i); v) - T(\mathcal{F}(i); w)|.$$

$$\tag{4}$$

To estimate g(i) we take a lemma from [6]. It is an easy exercise for the reader.

Lemma 10 If t_1, t_2, \ldots, t_m are non-negative real numbers and $t_1 + t_2 + \ldots + t_m \leq s$, then

$$\min_{1 \le j < \ell \le m} |t_j - t_\ell| \le \frac{s}{\binom{m}{2}}.$$

We distinguish two phases of the play.

Phase 1: $|V_i| = |V| - 2i > 2||\mathcal{F}||$. (Note that Beck uses $|V_i| > ||\mathcal{F}||$.) Simple counting shows that

$$\sum_{v \in V_i} T(\mathcal{F}(i); v) \le ||\mathcal{F}|| T(\mathcal{F}(i)).$$

By Lemma 10 and (4),

$$g(i) \leq \frac{||\mathcal{F}||}{\binom{|V_i|}{2}}T(\mathcal{F}(i)),$$

so by (3),

$$T(\mathcal{F}(i+1)) \le T(\mathcal{F}(i)) \left\{ 1 + \frac{||\mathcal{F}||}{\binom{|V_i|}{2}} \right\}.$$

Since $1 + x \le e^x = \exp(x)$, we have

$$T(\mathcal{F}(i+1)) \le T(\mathcal{F}) \exp\left\{ ||\mathcal{F}|| \sum_{j=0}^{i} \frac{1}{\binom{|V_j|}{2}} \right\}.$$

It is easy to see that

$$\sum_{i:|V_i|>2||\mathcal{F}||}\frac{1}{\binom{|V_i|}{2}} < \frac{1}{2||\mathcal{F}||},$$

so if i_0 denotes the last index of the first phase then

$$T(\mathcal{F}(i_0+1)) < \sqrt{e}T(\mathcal{F}). \tag{5}$$

Phase 2: $|V_i| = |V| - 2i \le 2||\mathcal{F}||$.

Then a similar counting as in *Phase 1* gives

$$\sum_{v \in V_i} T(\mathcal{F}(i); v) \le \left\lceil \frac{|V_i|}{2} \right\rceil T(\mathcal{F}(i)).$$

One checks that $T(\mathcal{F}(i+1)) \leq T(\mathcal{F}(i))$ when $2 \leq |V_i| \leq 4$. If $|V_i| \geq 4$, then by Lemma 10 and (4),

$$g(i) \le \frac{1}{|V_i| - 1} T(\mathcal{F}(i)),$$

so by (3),

$$T(\mathcal{F}(i+1)) \le \frac{|V_i|}{|V_i| - 1} T(\mathcal{F}(i)).$$
(6)

Let us recall the well-known Wallis' formula, $\lim_{n\to\infty}\frac{1}{2n+1}\prod_{i=1}^{n}\frac{(2i)^2}{(2i-1)^2} = \frac{\pi}{2}$. Since $\frac{(2n+2)^2}{(2n+1)(2n+3)} > 1$ for all $n \in \mathbb{N}$, we have the inequality for all $n \in \mathbb{N}$

$$\prod_{i=1}^{n} \frac{2i}{2i-1} < \sqrt{\frac{\pi}{2}(2n+1)}.$$
(7)

By repeated application of (6) we have

$$T(\mathcal{F}(end)) \le T(\mathcal{F}(i_0+1))2 \prod_{i:2 \le |V_i| \le 2||\mathcal{F}||} \frac{|V_i|}{|V_i| - 1} \le T(\mathcal{F}(i_0+1))2 \prod_{j=2}^{||\mathcal{F}||} \frac{2j}{2j - 1}.$$

Now using (7), (5) and (2), we have

$$T(\mathcal{F}(end)) < T(\mathcal{F}(i_0+1))\sqrt{\pi(||\mathcal{F}|| + \frac{1}{2})} \le \sqrt{e\pi}T(\mathcal{F})\sqrt{||\mathcal{F}|| + \frac{1}{2}} < 1.$$

That is, Chooser cannot completely occupy a winning set, and Theorem 5 follows. $\hfill \Box$

Acknowledgment. We would like to thank József Balogh for helpful comments. We are indebted to the referees for their numerous suggestions which improved the presentation of the paper.

References

- L. V. Allis, H. J. van den Herik and M. P. Huntjens, Go-Moku solved by new search techniques. Proc. 1993 AAAI Fall Symp. on Games: Planning and Learning, AAAI Press Tech. Report FS93-02, pp. 1-9, Menlo Park, CA.
- [2] J. Balogh, R. Martin and A. Pluhár, The diameter game. Horizon of Combinatorics, Balatonalmádi (2006).
- [3] J. Beck, Van der Waerden and Ramsey games. Combinatorica 1 (1981), 103–116.
- [4] J. Beck, Remarks on positional games. Acta Math Acad Sci Hungar 40 (1982), 65–71.
- [5] J. Beck, Deterministic graph games and a probabilistic intuition. Combinatorics, Probability and Computing 3 (1994), 13–26.
- [6] J. Beck, Positional games and the second moment method. Combinatorica 22 (2) (2002) 169–216.
- [7] J. Beck, Positional Games. Combinatorics, Probability and Computing 14 (2005), 649–696.
- [8] M. Bednarska, and T. Łuczak, Biased positional games for which random strategies are nearly optimal, *Combinatorica* 20 (2000), 477–488.
- [9] E. R. Berlekamp, J. H. Conway and R. K. Guy, Winning Ways for your mathematical plays, Volume 2, Academic Press, New York 1982.
- [10] A. Csernenszky and A. Pluhár, On the Complexity of Chooser-Picker Positional Games. In preparation.
- [11] P. Erdős and J. L. Selfridge, On a combinatorial game. J. Combinatorial Theory Series A 14 (1973), 298–301.

- [12] R. K. Guy and J. L. Selfridge, Problem S.10, Amer. Math. Monthly 86 (1979); solution T.G.L. Zetters 87 (1980) 575-576.
- [13] A. W. Hales and R. I. Jewett, Regularity and positional games. Trans. Amer. Math. Soc. 106 (1963) 222–229.
- [14] M. Krivelevich, Positional games and probabilistic considerations, Oberwolfach Report 20/2007 16–17.
- [15] A. Lehman, A solution of the Shannon switching game. J. Soc. Indust. Appl. Math. 12 1964 687–725.
- [16] J. G. Oxley, Matroid Theory. New York: Oxford University Press, 1992.
- [17] A. Pluhár, Generalized Harary Games. Acta Cybernetica 13 no. 1, (1997) 77–83.
- [18] T. J. Schaefer, On the complexity of some two-person perfect-information games. J. Comput. Syst. Sci., 16 (1978) no. 2, 185–225.
- [19] N. Sieben, Hexagonal polyomino weak (1,2)-achievement games. Acta Cybernetica 16 (2004), no. 4, 579–585.
- [20] H. Steinhaus, The problem of fair division, *Econometrica* **16** (1948) 101–104.