The Recycled Kaplansky's Game *

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Abstract

Motivated by the Nine Men Morris game, the achievement or hypergraph games can be prolonged in the following way. After placing a prescribed number of stones, the players pick some of these up and replace again. We study the effect of this recycling for the k-in-a-row game and some versions of the Kaplansky's game.

Keywords: positional games, achievement games, hypergraphs, Kaplansky.

1 Introduction and Results

A large number of combinatorial games were created from the earliest civilizations up to now; the authors of [7] try the impossible task of introducing a fraction of these. In a fascinating class of those, two players, I and II (later on M and B), put marks or move pieces on a board, while the outcome of the game depends on achieving certain geometrical configurations. The most prominent examples are the ageless Tic-Tac-Toe, the Nine Men's Morris, the Go-moku or its western variant, the 5-in-a-row.

Plenty of interesting games are relatively young, such as the Hex, Bridgit, Shannon's switching game or the Hales-Jewett games. In the case of the so-called *positional* or *achievement games* the rules can be unified. Given a finite or infinite set X (the "board"), the players alternately take elements of X (by marking or putting pieces onto it physically), and there is a fixed $H \subset 2^X$, the *winning sets*. A player wins by taking all the elements of a winning set first. For this sub-class we have a rich and beautiful theory.

Sometimes the players take p and q elements of X in turns, respectively. If $p \neq q$, it is a *biased* game, otherwise it is called *accelerated*, see [4, 5, 6, 10, 11, 12]. Since I always wins or the game is a draw when $p = q$ (see [7]), it also interesting to consider the *strong* or *Maker-Breaker version* of a game. Here Maker (I) wins by occupying a winning set, while Breaker (II) wins not by occupying such a set, but preventing Maker of doing so.

However, this pattern does not fit for such games as the recently solved Connect-4 or Nine Men's Morris, see [1], [2]. In the first case the available moves are restricted, while the whole static approach of the positional games is abandoned in the second.

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We shall address the issue of the second one and make an attempt to capture the idea of movements for a game. For an arbitrary positional game let us define the rules of the *recycled versions* as follows. For a natural number n the players make the first n steps as before; this is the *first phase*. Then, in the *second phase*, they just make moves with some of their earlier placed pieces in turns, instead of introducing new ones.

In order to investigate the effect of recycling, let us define some games. The first is the well-known *k-in-a-row* game ($k \in N$), which is played by the two players on the infinite (chess)board, or graph paper. They alternately put their own marks to previously unmarked squares, and whoever gets k-consecutive marks first (horizontally, vertically or diagonally) of his own, wins.

An interesting way to alter the *k-in-a-row* game is to relax the consecutiveness condition. We shall call the game $L_k(p,1;n)$ (or *line game* for short) for which:

1. I and II mark p and 1 squares in every step, respectively.

2. I wins upon getting k, not necessary consecutive, marks in a line (horizontally, vertically or diagonally), which is free of II's marks.

3. the game terminates after n steps

Then let $RL_k(p,1;n)$ be the recycled version of $L_k(p,1;n)$.

Our third subject is the Kaplansky's game, where the players put their marks on the Euclidean plane. Here I wins achieving k marks on a line, provided II has no mark on that line. Now $K_k(p,q)$ stands for the version in which I and II marks p and q points, respectively. Let $K_k(p,q;n)$ be the version which ends after n round, and $RK_k(p,q;n)$ be its recycled version.

Before stating our theorems, let us recall some earlier results on these games.

The *recycled k-in-a-row* (no matter when does the second phase start) turns out to be easy, because the decomposition methods utilized in [7] still work, and give the same bounds. That is even the Maker-Breaker version of the recycled k-in-a-row game is a draw if $k \geq 8$.

Bounds for the games $L_k(p,1;n)$ and $RL_k(1,1;n)$ are less obvious, we shall prove:

Theorem 1 *In the Maker-Breaker $L_k(p,1;n)$ game, Breaker wins if $k \geq p\log_2 n + p\log_2 p + 3p$. On the other hand, Maker wins if $p > 1$ and $k \leq c\log_2 n$ for some $c > 0$.*

Theorem 2 *Breaker wins the Maker-Breaker $RL_k(1,1;n)$ game if $k \geq 32\log_2 n + 224$.*

In the version of Kleitman and Rothschild (see in [3]) I (II) wins by getting k (l) points of a line while the opponent has none of that line, respectively. They prove that, given any $k \geq 1$, there is an $l(k)$ such that II has a winning strategy whenever $l \geq l(k)$. Beck in [4] considers little different games; here I wins with k points on a line, II with l, and I may mark p points on each turn, while II only one per turn. Here II wins if $l < ckp\log(p+1)$, for some $c > 0$. He has also shown there exist $C > c > 0$, such that in $K_k(1,1;n)$: Maker wins if $k < c\log_2 n$ and Breaker wins if $k > C\log_2 n$. For its recycled version we have the following result.

Theorem 3 *Breaker wins the Maker-Breaker $RK_k(1,1;n)$ game if $k > cn^{1/3}$.*

2 Proofs

2.1 Weight functions

In the proof of the Theorems 1, 2 and 3 we heavily use the *weight function method*, which was developed in [5] and developed in [6] and [8]. First let us recall some earlier definitions and results.

A pair of (X, H) is called a *hypergraph* if $H \subset 2^X$. If (X, H) is a hypergraph, then a $(p, q, H) - game$ (or simply hypergraph game) is a game in which I selects p and II select q previously unselected elements of X. The first, who takes all elements of an $A \in H$, wins. A (p, q, H)-game has a so-called *Maker-Breaker version* in which I wins taking an edges of the hypergraph any time. One of the most important result on such games is the Erdős-Selfridge theorem; one of its generalization is due to József Beck.

Theorem 4 ([5]) *Breaker wins the $(p, 1, H) - game$ if* $\sum_{A\in H} 2^{-\frac{|A|}{p}} < \frac{1}{2}$.

In our cases this theorem cannot be applied directly, since the hypergraphs involved are infinite, and it is not known if Theorem 4 holds for recycled games. The following lemma is also due to Beck (see [5]). We repeat the proof in order to see the properties of the used weight function.

An edge $A \in H$ is *active* if Breaker has not taken any of its elements.

Lemma 1 *Playing a $(p, 1, H)$ game, Breaker can assure that no active edge contains more than $p + p \log_2 |H|$ elements taken by Maker.*

Proof of Theorem 4. We may assume Maker starts the game. For any $A \in H$ let $A_k(M)$ and $A_k(B)$ be the number of elements in A, after Makers kth move, selected by Maker and Breaker, respectively. Now, for an $A \in H$

$$w_k(A) = \begin{cases} \lambda^{A_k(M)} & \text{if } A_k(B) = 0 \\ 0 & \text{otherwise} \end{cases}$$

where $\lambda > 0$, and for any $x \in X$ let $w_k(x) = \sum_{x\in A} w_k(A)$. The numbers $w_k(A)$ and $w_k(x)$ are called the *weight* of A and x (in the kth step), respectively. When it does not cause confusion we may suppress the lower index.

Now selecting an element in the kth step Breaker uses the *greedy* algorithm, i.e. chooses an unselected element $y^k \in X$ of maximum weight. Let $x_1^{k+1}, ..., x_p^{k+1}$ be the elements selected by Maker in the $(k+1)$st step and $w_k = \sum_{A\in H} w_k(A)$ be the *total sum* or *potential.* For $k \geq 0$, following inequality holds for the potential:

$$w_k - w_k(y^k) + (\lambda^p - 1)w_k(y^k) \geq w_{k+1}.$$

Indeed, w_k decreases by $w_k(y^k)$ upon selecting y^k. The elements selected by I in the $(k+1)$st step cause the biggest increase if $w_k(x_l^{k+1})$ is maximal for $1 \leq l \leq p$, and for all A such that $w_k(A) \neq 0$ we have $x_l^{k+1} \in A$ iff $x_m^{k+1} \in A$, $1 \leq l,\ m \leq p$. Since the increase in this case is just $(\lambda^p - 1)w_k(y^k)$, the inequality is proved. Setting $\lambda = 2^{1/p}$, we get $w_k \geq w_{k+1}$, $k \geq 0$, which justifies that w_k is called potential.

Particularly $w_1 \leq (\lambda^p - 1)|H| + |H| \leq 2|H|$. Since $q = 1$ and the elements of H are the same size, the inequality $\sum_{A \in H} 2^{-|A|/p} < 1/2$ leads to the inequality $2|H| < 2^{|A|/p}$. Assume that Maker wins the game in the kth step. This would imply $w_k \geq \lambda^{|A|} = 2^{|A|/p}$, which contradicts the monotonicity of the potential. □

Proof of Lemma 1. Just take the logarithm of the inequality $\lambda^{A_k(M)} = w_k(A) \leq w_k \leq w_1 \leq 2|H|$ that holds for any active edge $A \in H$. □

2.2 Proof of Theorem 1

Let us recall that a line L means consecutive squares along an infinite line here (horizontally, vertically or diagonally). Now we have infinitely many interacting sets, so the weight function method does not seem to be helpful. The way to overcome the difficulties is to change the definition of the weights. The price of this is that the potential is no longer a decreasing function, but an increasing one. However, we can control the growth, since the game lasts only n steps.

Let H be the set of all lines, and $L_j(M)$ and $L_j(B)$ the number of squares of line L marked by Maker and Breaker after the jth step, respectively. Now the weight function of L at the jth step:

$$w_j(L) = \begin{cases} \lambda^{L_j(M)} \text{ if } L_j(M) \geq 1 \text{ and } L_j(B) = 0 \\ 0 \qquad\qquad\qquad\qquad\quad \text{otherwise} \end{cases}$$

where $\lambda = 2^{\frac{1}{p}}$.

For a square q,

$$w_j(q) = \sum_{L \in H, q \in L} w_j(L)$$

is the weight of q, and

$$w_j = \sum_{L \in H} w_j(L)$$

is the *total weight* at the jth step.

Breaker applies the greedy selection. For the weight functions, similarly to the proof of Theorem 4, we have

$$\sum_{L_j(M) \geq 1} w_{j+1}(L_{j+1}) \leq \sum_{L \in H} w_j(L_j).$$

On the other hand, in each step the number of lines whose weight becomes positive is at most $4p$, and the weight of such a line is no more than $\lambda^p = 2$. That is

$$w_{j+1} \leq w_j + 8p$$

holds for $0 \leq j \leq n$, where $w_0 = 0$. That is if the line L is unblocked at step j (i.e. $L_j(B) = 0$) and $L_j = i$ than

$$\lambda^i \leq 8pj \Leftrightarrow i \leq p(\log_2 j + \log_2 p + 3).$$

Since $0 \leq j \leq n$, the first part of Theorem 1 follows.

The second part is fairly standard, we give just the sketch of its proof. In fact, one (say vertical) winning direction is enough. Maker divides the game into phases. For the sake of simplicity we omit to write the integer parts. In the first phase Maker places $n(p-1)/p$ element in a row. Call a column *i-free* if it contains i marks of Maker, but none of Breaker. At the end of the first phase the number of 1-free columns is at least $n((p-1)/p)^2$. In the ith phase Maker uses up $n((p-1/p))^i$ new mark, each is placed to an $i-1$-free column. It is easy to check that Maker can reach the ith phase if $n((p-1)/p)^i \geq 1$, and uses up at most n marks. That is an i-free column appears if $i \leq c \log_2 n$, where c is about $(\log_2 p - \log_2(p-1))^{-1}$. □

2.3 Proof of Theorem 2

Breaker divides the game into sub-phases. The first sub-phase is the first phase of the game, then a sub-phase consists of n pair of moves. Defining the weight function as before, but $\lambda = \sqrt{2}$, Breaker places every second mark (the *active* marks) according to the greedy strategy and deposits the others arbitrarily, i.e. *in reserve*). It may happen that one of Breakers reserved marks is already on the square q, which is to be occupied by an active mark of Breaker. In that case Breaker places the new mark arbitrarily (sends it into reserve), and the mark on the square q becomes active.

Considering only the effect of Breakers active marks, the game reduces to the game $L_k(2,1,n)$. That is Lemma 1 applies, and for any line L if $L_j(M) = i$ and $L_j(B) = 0$, then $i \leq 2(\log_2 j + 4)$ if $0 \leq j \leq n$.

In the other sub-phases Breaker plays a fictitious game, and keeps the status of his marks (active or reserved) strictly. The marks of Maker are indexed by the numbers $1, 2, \ldots, n$. At the beginning of a sub-phase Breaker cannot see Makers marks, and in the jth step Makers new mark and the mark indexed by j become visible for Breaker as new moves. (If Maker moved the jth mark, only one mark becomes visible.)

However Breaker responds only in every second step, using the marks from the reserve. (Breaker does nothing in the odd steps. If picking up a mark and putting back to the same place is permitted, it is easy. If it is not, Breaker designates a mark at the very beginning, which is neither active nor reserved, and moves this mark arbitrarily in the odd steps.)

Trying the previous greedy strategy another difficulty arises. Breaker may not occupy the square q of maximum weight because q has been already taken (by one of Makers invisible marks or one of Breakers own reserve). Then, Breaker blocks the lines going through q, using four marks. (See a similar idea in [12].) Now, looking only Makers visible marks, if for a line L, $L_j(M) = i$ and $L_j(B) = 0$ then $i \leq 16(\log_2 j + 7)$, since after at most 16 moves of Makers, Breaker may reply, and Theorem 1 applies.

By the end of a sub-phase Makers all marks become visible, and a line L, which contain more than $16(\log_2 n + 7)$ of them, is blocked by Breakers reserve. Finally, Breaker starts the next sub-phase renaming his marks, the active ones become reserved and vice versa.

Since the active marks control the invisible marks during a sub-phase, if for a line L the sum of visible and invisible marks of Maker on L is i, and L is not blocked (by the active marks or by the reserve), then $i \leq 32(\log_2 n + 7)$. □

2.4 Proof of Theorem 3

The most natural idea is to mimic the proof of Theorem 2.

Unfortunately it breaks down irreparably at the point where Breaker wants to occupy, or at least block the point q, which is already taken. The problem is that q can be the element of many lines, so Breaker cannot cancel the weight of q by using only constantly many points.

To overcome this difficulty, we change the weight function and give a more sophisticated analysis of it.

Let the weight of a line L after Maker jth move be

$$w_j(L) = \begin{cases} \lambda^{L_j(M)} \text{ if } L_j(M) \geq c_1 n^{1/3} \text{ and } L_j(B) = 0 \\ 0 \qquad\qquad\qquad\qquad\qquad\qquad \text{otherwise} \end{cases}$$

where $\lambda = \sqrt{2}$ and $c_1 > 0$ will be specified later.

As before, for a point x, $w_j(x) = \sum_{L \in H, x \in L} w_j(L)$ is the weight of x, and $w_j = \sum_{L \in H} w_j(L)$ is the *total weight* at the jth step.

However, Breaker uses not only the greedy strategy, the recycled point also have to be designated. When Breaker removes a point y, the total weight function may grow. It grows iff there is a line L containing y such that $L_j(M) \geq c_1 n^{1/3}$ and $L_j(B) = 1$. Obviously the number of such points cannot be bigger than the number of lines containing at least $c_1 n^{1/3}$ points of Maker. To estimate this, we need a definition and a theorem of Szemerédi and Trotter.

An *incidence* of a point and a line is a pair (p, L), where p is a point, L is a line, and p lies on L.

Theorem 5 ([14]) *Let I denote the number of incidences of a set on n points and m lines. Then $I \leq c(n + m + (nm)^{2/3})$.*

Let us note that László Székely published a new, more accessible proof of Theorem 5, see in [13].

An easy corollary of Theorem 5 is that there is a constant c_2 such that the number of lines containing at least k points of S is less than $c_2 n^2/k^3$ whenever $k \leq \sqrt{n}$.

That is if $c_1 > c_2^{1/3}$, then the number of lines containing at least $c_1 n^{1/3}$ points of Maker is less than n. It means Breaker can always find a mark y such that its removal does not affect the value of the total weight function. The steps of Maker and Breaker are $x_1, x_2, \ldots x_i$ and $y_1, y_2, \ldots y_i$, respectively.

As before, for the weight function we have

$$w_{j+2} \leq w_j - w_j(y_j) - w_{j+1}(y_{j+1}) + w_j(x_{j+1}) + w_{j+1}(x_{j+1}) + \frac{2}{c_1} n^{2/3} \lambda^{n^{1/3}} \lambda^{n^{1/3}+1}.$$

Here the term $f(n) := \frac{2}{c_1} n^{2/3} \lambda^{n^{1/3}} + \lambda^{n^{1/3}}$ bounds the growth caused by the lines that of weight becoming positive in the jth and $(j+1)$th steps. By the argument of Theorem 4, $w_j(y_j) \geq w_j(x_{j+1}) + w_{j+1}(x_{j+1})$, since $\lambda = \sqrt{2}$. We also have $w_{j+1}(y_{j+1}) > w_{j+1}/n$, since the number of positive weighted lines is less than n, giving

$$w_{j+2} \leq w_j - \frac{w_{j+1}}{n} + f(n).$$

On the other hand, $w_{j+2} \le w_{j+1} + f(n)$, or equivalently $w_{j+1} \ge w_{j+2} - f(n)$. That is the value of w_{j+2} is bounded, since if $\frac{w_{j+1}}{n} \ge f(n)$, and then we have $w_{j+2} \le w_j$. From here one gets that $w_{j+2} \le (n+1)f(n)$. It means that if for a line L, $L_{j+2}(M) = s$ and $L_{j+2}(B) = 0$, then $(n+1)f(n) \ge w_{j+2} \ge \lambda^s$. Taking the logarithm of both sides, $s \le 2\log_2 w_{j+2} \le 2n^{1/3}$, provided n is big enough. □

2.5 Remarks and Open Questions

As we have seen, there is a large gap between the logarithmic lower and $O(n^{1/3})$ upper bound what Maker can achieve in the recycled Kaplansky's game.

Question 1 *Can the upper or lower bounds of Theorem 3 improved?*

Even less is known about recycled hypergraph games in general. It is easy to give example for which Breaker wins the first phase of the game, while Maker wins the recycled version.

Question 2 *Is there a hypergraph game won by Breaker, but Maker wins its recycled version?*

It is also interesting if the Erdős-Selfridge theorem extends to the recycled games.

Question 3 *Is it true if $\sum_{A\subset H} 2^{-|A|+1} < 1$, then Breaker wins the recycled version of the (X, H) game?*

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