Problem Sheet to Combinatorial Games

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"A beginning is the time for taking the most delicate care that the balances are correct."

Frank Herbert, Dune

1 The Basics

Exercise 1.1 (NIM) *The game of NIM is played on heaps of pebbles, and the players (first, second) may remove pebbles from one heap by turns. (They have to take at least one pebble, though.) The player who takes the last pebble, wins the game. C. L. Bouton proposed the following sizes for heaps:* 8, 7, 6, 5, 4, 3, 2 *and* 1.

- *Show that with this setup the first player wins.*
- *What is the first move of the first player?*

Exercise 1.2 (Rook game) *We have an* 8×8 *chessboard with eight white rooks and eight black rooks on the first row and the eight row, respectively. White and Black (players), starting with White, are taking turns, and they may play by the following rules:*

- *The rooks may move only up and down.*
- *They cannot capture.*

The player, who cannot move in his own turn, loses.

- *Show that White wins.*
- *White can win in at most* 48 *steps.*

Exercise 1.3 (Divisor game) *Given a positive natural number* n*. First and Second have to choose among the divisors of* n *alternately in such a way, that the multiples of a previously chosen number may not be taken. Who takes one, loses the game. Show that First wins.*

Problem 1.1 (Gale's game) *Given a finite set H, First and Second alternately select its proper subsets (i. e. taking H itself or the $\emptyset$ is illegal). Furthermore they cannot take subsets of sets taken earlier. Who wins tha game?*

Exercise 1.4 (Tic-Tac-Toe) *First and Second alternately put a token onto a square of a 3×3 board. Whoever occupies a winning set (full row, column or main diagonal) first, wins the game. The are modified version of Tic-Tac-Toe. In the* **Maker-Breaker** *version Maker wins occupying a winning set (no matter when), while Breaker wins if he can prevent this. In* **misere** *or* **reversed** *Tic-Tac-Toe the first occupation of a winning set loses. Finally in the* **moving** *version takin a winning set is the goal again, but the players have only $3-3$ tokens, after using these up they take turns, and they have to move one of they own tokens to a neighboring (by an edge) empty square. The followings hold:*

- *The $Tic-Tac-Toe$ game is a tie.*
- *Maker (First) wins the Maker-Breaker version. What if he is the second?*
- *The reverse Tic-Tac-Toe is a tie.*
- *First wins the moving version.*

Exercise 1.5 (Lewthwaite's game) *Let us place white and black stones to the white and black squares of a 5×5 chessboard, respectively, except the middle square, which is left empty. Then White and Black alternately push one of they own stones to the (edge) neighboring empty square. If a player cannot move in his turn, loses. Show that White wins the game.*

Exercise 1.6 (Replicating stones) *This is a solitary game (= there is only one player). Consider the first quadrant of the coordinate system as an infinite chessboard (its rows are numbered by 1, 2, 3,... while its files may referred as a, b, c,...). We place a stone onto the square $a1$, which may be removed on the price that we are bound to place its two successors onto $a2$ and $b1$. Generally a stone can be removed from a square if its upper and right neighbors are stone-free, and we have to place 1-1 stones onto these squares. The goal of the game is to make the squares $a1$, $a2$, $a3$, $b1$, $b2$, $c1$ free of stones. Show that it is impossible.*

Exercise 1.7 (Perfect information, finite, two-person games) *Give a unified proof that in the previous examples (except example 1.6) out of the following statements exactly one holds:*

- *First (White, Maker) wins.*
- *Second (Black, Breaker) has at least a tie.*

This is the so-called game theoretic **tertium non datur** *("the third case does not exist" principle). Note that it may not hold in infinite games.*

2 Matchings and Graphs

Exercise 2.1 (Strong connectedness) *Show that*

- *The $V(G)$ vertex set of a directed graph G is the disjoint union of its components (i. e. the strongly connected subgraphs).*
- *In the previous decomposition the edges between two components go in one direction.*
- *The $C(G)$ condensation of G is the following. The vertices of $C(G)$ are the components of G, and we draw one directed edge between two vertices, if there are edges between the corresponding components in G. (The direction is also preserved.) $C(G)$ does not contain directed cycles.*
- *For a G undirected graph $\chi(G) \leq 2$ if and only if G does not contain any odd circuits.*
- *For a G directed graph $\chi(G) \leq 2$ if and only if G does not contain any odd (directed) cycles.*

Exercise 2.2 (Kernel of graphs) *A directed graph G is* **symmetric** *if for any vertices $x, y \in V(G)$ hold that $(x, y) \in E(G)$ iff $(y, x) \in E(G)$.*

- *If the graph G is symmetric, then it has a kernel.*
- *If G is a cycle-free directed graph, then G has exactly one kernel.*
- *If G is a directed graph containing no odd cycles, the G has a kernel.*
- *Draw a graph G having no kernel.*

Exercise 2.3 (Generalized Slither) *Given a directed graph G, and the first player chooses a starting vertex. Then Second and First alternately choose among the yet untaken vertices, such that if the last step of the opponent is x_i and the new step is x_{i+1}, then $(x_i, x_{i+1}) \in E(G)$. The player, who is unable to move, loses.*

- *If G is cycle-free, then the first player wins. From which starting points can he win?*
- *Assume that there is an $x_0 \in V(G)$ for which $d_-(x_0) = 0$, i. e. no edges go to x_0. Then First has a winning strategy.*
- *Second wins for a directed graph G iff he wins the generalized Slither for all strongly connected component of G.*
- *For a symmetric graph G First wins iff there is no perfect matching in G.*

Exercise 2.4 (Dirac condition) *Let us show that*

- *If G is an undirected graph on $2k > 0$ vertices, and for all $x \in V(G)$ $d(x) \geq k$, then G has perfect matching.*
- *There is even a $2k$ lenght (Hamiltonian) circuit in G.*

Exercise 2.5 (Silverman's game) *The game needs an $n \times n$ chessboard, and the players has 1-1 piece, that is called* **amazon**. *The legal movements of a amazon is the union of the queen and the knight of the regular chess pieces. The first player (White) put his pieces on the board, then the second player (Black) does the same such that the two pieces could not take each other. Then, taking turns, they replace their pieces marking the squares they occupied before. The replecement has to rules; the piece cannot be put onto a marked square, or to the capture range of the opponent's piece. The player, who cannot make legal move in his turn, loses the game.*

- *If $n \leq 5$, then White wins.*
- *If $n \geq 6$ and n is even, then Black wins.*
- *If $n \geq 7$ and n odd, then White wins.*

Exercise 2.6 (Bandwidth) *The squares of an $n \times n$ board are marked by the numbers from 1 to n^2. Two squares are neighbors, if they meet in an edge. Show that there must be two neighbors such that the difference of the numbers written on those is at least n. (A minimum of the maximal difference is called the* **bandwidth** *of the adjacency graph.)*

Exercise 2.7 (Bandwidth game) *First and Second alternately write numbers to clear squares of an $n \times n$ board from the set $1, ..., n^2$, using a number only once. The first player, who writes down a number k to a square s such that one of s neighbors has already marked by a number which difference from k is at least n, loses.*

- *If n is odd, then First wins.*
- *If n is even, then Second wins.*

A possible Maker-Breaker and reverse version is to fix an $m > n$, and Maker wins, if Breaker creates a difference at least m, otherwise loses.

Problem 2.1 (Bandwidth for triagles) *It is know that the bandwith of T_n (the regular triangulation of a triagle, where n stand for the number of points on a side) is n. What can one say about the corresponding game?*

Exercise 2.8 (Black's game) *We start to draw a line at the Nort-West corner of a 5×5 board. The player alternately put one of the three tile in such a way that the line should be lenghten. The player, who hit the border, loses. (The line otherwise may cross itself, and may use up part already drawn to the board.) Show that First wins.*

3 Hypergraph Games

Definitions.

Given a set X and a set $H \subset 2^X$, then the pair (X, H) is a *hypergraph*. X is its *groundset*, or set of *points*, while the element of H are the *edges*.

Let X be an arbitrary set, $H_1, H_2 \subset 2^X$, and p and q natural numbers.

The (X, H_1, H_2, p, q) *hypergraph game* or shortly the (X, H_1, H_2, p, q) *game* is the following.

First and Second alternately takes p and q previously untaken element of X, respectively. First wins getting all elements of an $A \in H_1$, if meantime Second has not taken all elements of a $B \in H_2$. Similarly, Second wins getting a whole edge $B \in H_2$, if First has not taken yet a whole edge from H_1.

Instead of the clumsy (X, H_1, H_2, p, q) notion we may use (X, H), when $p = q = 1$ and $H_1 = H_2 = H$, the (H, p, q), when $H_1 = H_2 = H$, furthermore the (X, H_1, H_2) notion, when $p = q = 1$ hold.

The generalization of Tic-Tac-Toe (see exercise 1.4) are handy on the structure (X, H_1, H_2, p, q), too. The *Maker-Breaker* (X, H) hypergraph game is that in which Maker wins occupying a whole edge from H, while Breaker wins, if he can prevent this. The reverse (misere) version is also straigthforward.

Exercise 3.1 (Easy) *Show that*

- *The (X, H_1, H_2, p, q) game may be a tie.*
- *Most of the previous games are hypergraph games.*
- *Hypergraph games can be viewed as Conway games, or generalized Slither. (This does not help much, though.)*

Exercise 3.2 (Examples) *Write tic-tac-toe-t, 5-in-a-row, hex, bridgit and the Shannon switching game as $(X, H_1, H_2, 1, 1)$ or Maker-Breaker (X, H) game.*

Exercise 3.3 (Reduction) *Show that a Maker-Breaker (X, H, p, q) game can be considered as a (X, H_1, H_2, p, q) hypergraph game, which cannot end with tie.*

Exercise 3.4 (Strategy stealing for the (X, H) game) *Show that First wins in the (X, H) game, or it is a tie. What happens in the reverse (X, H) game?*

Exercise 3.5 (Chromatic number) *We want to color the X groundset of (X, H) such that every edge should contain at least two different colors. The minimal number of color is the chromatic number of the hypergraph, denoted by $\chi(H)$. Show that*

- *If $\chi(H) \geq 3$, then the (X, H, p, q) game cannot be a tie.*
- *If $\chi(H) \geq 3$, then First wins the (X, H) game.*

Exercise 3.6 (Hex) *Show that*

- *There is no tie in hex, played on the $m \times n$ board.*

- *First wins in hex played on $n \times n$ board.*
- *Hex played on the $n \times (n+1)$ board is a win for the player, who has to bridge the shorter distance.*

Exercise 3.7 (5-in-a-row, k-in-a-row) *The following holds for the k-in-a-row*

- *Second cannot win the game k-in-a-row for any $k \in N$.*
- *First wins the game k-in-a-row, if $k = 1, 2, 3, 4$ or 5.*
- *The game k-in-a-row is a tie, when $k \geq 9$.*
- *The exist a matching strategy for Second resulting in a tie, when $k \geq 9$.*
- *The game k-in-a-row is a tie, when $k \geq 8$.*

Exercise 3.8 (Trees) *For a graph G on n vertices the followings are equivalent*

- *G is a* **tree**, *i. e. connected and circuit-free.*
- *G is connected, but removing any of its edge e the remaining graph $G \setminus e$ is not connected any more.*
- *G is circuit-free, but adding an extra edge e to $E(G)$ a circuit is created.*
- *G is circuit-free and has $n-1$ edge.*
- *G is connected and has $n-1$ edge.*

Exercise 3.9 (Shannon switching game=SSG) *Show that*

- *For a graph G Second (as Maker) wins the SSG iff G contains two disjoint spanning trees.*
- *Bridgit can be reduced to SSG.*
- *Let G be is the disjoint union of two trees. How to play the reverse SSG?*
- *First can win Bridget utilizing a a matching strategy.*
- *Construct a graph G for which Second (Maker) wins SSW, but there cannot be winning matching matching strategy.*
- *An (X, H) hypergraph is the base set of a* **matriod**, *if*

  1. $\forall\ A, B \in H$ *hold that* $A \not\subset B$
  2. $\forall\ A, B \in H, \forall\ x \in A \setminus B\ \exists\ y \in B \setminus A$, *such that* $\{A \setminus \{x\}\} \cup \{y\} \in H$.

  Suppose that Breaker starts a Maker-Breaker game on (X, H). Then Maker wins iff there exist A and B elements of H such that $A \cap B = \emptyset$.

- *The previous is the generalization of SSG. Give hypergraph, whose edges are the bases of a matroid, but these edges do not correspond to the spanning trees of any graph G.*
- *Let us divide $V(G) = A \cup B$, and $A \cap B = \emptyset$. The edges of G between A and B are the A, B-cut. Breaker wins SSW for the graph G iff he can claim all edges of an A, B-cut.*

4 Matchings for Hypergraph Games

We need to recall Hales-Jewett's, van der Waerden's and Ramsey's theorems as well as the results Gyula König and Ph. Hall. (Out of these arise the $H(n,d)$, $W(N,n)$ and $R(N,n)$ games.)

Theorem 4.1 (Hales-Jewett) *For all natural number n there exist a $d > 0$ integer, such that $\chi(H) > 2$ for the hypergraph of the $HJ(n,d)$ game.*

Theorem 4.2 (van der Waerden) *For all natural number n there exist a $N > 0$ integer, such that $\chi(H) > 2$ for the hypergraph of the $W(N,n)$ game.*

Theorem 4.3 (Ramsey) *For all natural number n if $N > 4^n$, then $\chi(H) > 2$ for the hypergraph of the $R(N,n)$ game.*

A graph G **bipartite**, if $V(G)$ is the disjoint union of A and B such that $E(G) = (A, B)$-cut, that is all the edges of G are between A and B. An $F \subset E(G)$ is the matching of B to A if F is a matching and every vertex of B incident to an edge of F. For a graph G and $X \subset V(G)$ let $N(X)$ be the **neighborhood** of X, that is those vertices of G which are connected to at least one vertex of X. If for a bipartite graph G there exists an $X \subset B$, such that $|X| > |N(X)|$, then B obviously cannot have a matchingto A. König's theorem says the reverse is also holds.

Theorem 4.4 (König) *For a G finite bipartite graph, where $V(G) = A \cup B$, $A \cap B = \emptyset$, and A and B independent sets, B has a macthing to A iff for all $X \subset B$ the $|X| \leq |N(X)|$ holds.*

Philip Hall stated it slightly different form, for which we need a further definition. Let $\{A_i\}_{i=1}^m$ be a finite system of finite sets. A $S = \{s_1, ..., s_m\} \subset \cup_{i=1}^m A_i$ *system of disjoint representatives* (shortly SDR), if $s_i \neq s_j$ for $i \neq j$, and $s_i \in A_i$ for $i = 1, ..., m$.

Theorem 4.5 (Ph. Hall) *$\{A_i\}_{i=1}^m$ has* SDR *iff for all $I \subset \{1, ..., m\}$ the $|\cup_{i \in I} A_i| \geq |I|$ inequality holds.*

This form immediately gives the Hales-Jewett theorem.

Theorem 4.6 (Hales-Jewett) *If for a finite (X, H) hypergraph game for all $G \subset H$ the inequality $|\cup_{A \in G} A| \geq 2|G|$ holds, then the game is a tie.*

Exercise 4.1 (Chromatic numbers) *To make it easier, we denote the hypergraphs of the $R(N,n)$, $W(N,n)$ and $HJ(n,d)$ games by R, W and H, respectively. Show that*

- $\chi(R) \geq 3$, *if* $N > 4^n$.
- $\chi(R) \leq 2$, *if* $N < 2^{n/2}$.
- $\chi(W) \leq 2$, *if* $N < 2^{n/2}$.
- $\chi(H) = 3$, *if* $n = d = 3$.
- $\chi(H) = 2$, *if* $n = 3$ *and* $d = 4$.
- $|H| = \frac{1}{2}((n+2)^d - n^d)$.
- *In the hypergraph H every vertex is an element of at most $\frac{1}{2}(3^d - 1)$ edge, if n is odd. For even n, this number is $2^d - 1$.*
- $\chi(H) \leq 2$, *if* $n = 2l + 1$ *and* $n - 3 - \log_2 n > (\log_2 3)d$, *or if* $n = 2l$ *and* $n - 3 - \log_2 n > d$.
- $\chi(H) \leq 2$, $n \geq d + 1$.
- $\chi(R) = 3$, *if* $N = 6$ *and* $n = 3$.

- *$R_k(N,n)$ means such a Ramsey type game in which First and Second take k element subsets of a set of N element, and the player who takes all k element subset of a (sub)set of n elements first, wins the game. Show that First wins, if N is big enough.*

Exercise 4.2 (Matchings in graphs) *Show that*

- *If a graph G is a tree, then G is bipartite.*
- *A graph G is bipartite iff all of its circuits consists of even number of vertices.*
- *Let G be such a bipartite graph, in which $|A| = |B|$ and the degree of for every vertex is $k \geq 1$. Then G has a perfect matching.*
- *König's theorem equivalent to Hall's theorem, so the reference as* **König-Hall theorem** *is justified.*
- *There is a bipartite graph G, ($V(G) = A \cup B$) on a countably infinite set, which satisfies the condition of the König-Hall theorem, but there is no matching of B to A.*
- *Let G be a countably infinite bipartite graph satisfying the König-Hall condition for B-re, and the degree of all vertex is finite. Then there exits a matching of B.*

Exercise 4.3 (Hales-Jewett games) *Show that*

- *What is* $\mathrm{HJ}(3,2)$ *and* $\mathrm{HJ}(4,3)$*?*
- *Prove the Hales-Jewett theorem via Philip Hall's theorem.*
- *Maker wins the Maker-Breaker* $\mathrm{HJ}(n,d)$ *game, if* $n < \sqrt{\frac{2d}{\log 2}}$.
- *The game* $\mathrm{HJ}(n,d)$ *is a tie, if* $n \geq 3^d - 1$ *and* $n = 2l+1$, *or if* $n \geq 2^{n+1} - 2$ *and* $n = 2l$.
- *First (Second) does not lose the reverse* $\mathrm{HJ}(n,d)$, *when* d *is odd (even).*

Exercise 4.4 (k-in-a-row via Hales-Jewett and M. Hall theorem) *Using Marshall Hall's theorem (a exercise 4.2 item 6) show that*

- *There is a pairing strategy for Second resulting in a tie for the game* 14-*in-a-row.*
- *Generalize the above result for the* d*-dimensional* k*-in-a-row.*

Exercise 4.5 (Matching for reversed Hypergraph games) *Explore if there is a Hales-Jewett type of result for reverse hypergraph games.*

Exercise 4.6 (Tournaments 1.) *A* T_n **tournament** *(i. e. directed complete graph) is* **transitive**, *if its vertices can be ordered such that every edge goes from a vertex of smaller index to a vertex of higher index. First and Second taking turns mark and direct a previously untouched edge of* K_n. *The player who first builds up a transitive* T_n *tournament from the edges that he marked, wins the game.*

- *First wins, if* $N > 2^{4^n}$.
- *The game is a tie, if* $N < 2^{\frac{n}{2}-2}$.

5 Weigth functions

The use of weigth functions (also called potential functions, or sometimes conditional probability method) is very old. One finds it in game theory relatively late, e. g. Conway's frogs, or the celebrated Erdős-Selfridge theorem.

The idea in the use of weight function is to measure the "danger" that a winning set A becomes completely that of Maker during a hypergraph game. This is going to be a non-negative function that grows when Maker takes an element of A, and drops to zero if Breaker does that. We sum up the initial values, and show that this sum is monotone decreasing upon Breaker follows the greedy strategy; that is why we call this sum potetial. Since the win of Maker implies a bigger value of potential than it is possible, we conclude the game is in fact a tie.

Theorem 5.1 (Erdős-Selfridge) *If for a Maker-Breaker* (X, H) *game Maker starts, and*

$$\sum_{A\in H} 2^{-|A|+1} < 1,$$

then Breaker wins.

Proof. Let $A_i(I)$ and $A_i(II)$ be the number of elements taken from the set $A \in H$ by Maker and Breaker *after* the ith step of Maker and *before* the ith step of Breaker, respectively. Maker ith step is x_i, while that of Breaker is y_i. The *weight* of a set $A \in H$ (after the ith step of Maker, and before the step of Breaker) is

$$w_i(A) = \begin{cases} 2^{-|A|+A_i(I)} & \text{if } A_i(II) = 0 \\ 0 & \text{otherwise} \end{cases}$$

The *weight* of an $x \in X$ is $w_i = \sum_{x\in A} w_i(A)$, and the *total weight* or *potential* is $w_i = \sum_{A\in H} w_i(A)$. Breaker applies the *greedy strategy*; out of the elements still available, he takes the one of maximum value, denoted by $w_i(z)$. We claim this strategy results in the monotone decrease of w_i, i. e. $w_1 \geq w_2 \geq ... \geq w_i$. Indeed, after the ith step of Breaker the value of w_i goes down by $w_i(z) = w_i(y_i)$. The $(i+1)$st step of Maker may lift this value by at most $w_i(x_{i+1})$, since he simply doubles the weight of sets containing x_{i+1}, but not containing y_i. (The value of set containing y_i dropped to zero.) Because of the greedy selection $w_i(y_i) \geq w_i(x_{i+1})$, which gives the monotonicity of the potential. On the other hand

$$w_1 \leq \sum_{A\in H} 2^{-|A|+1} \text{ since } w_0 \leq \sum_{A\in H} 2^{-|A|}$$

and x_1 doubles the weight of a set A for which $x_i \in A$. In the worst case for all $A \in H$, and then $w_1 = 2w_0 = \sum_{A\in H} 2^{-|A|+1}$.

Let us suppose that Maker wins the game in the ith step. That means there exists an $A^* \in H$, such that $|A^*| = A_i^*(I)$, implying

$$w_1 \geq w_i = \sum_{A\in H} w_i(A) \geq w_i(A^*) = 2^{-|A^*|+A_i^*(I)} = 2^0 = 1.$$

But we have seen $w_1 \leq \sum_{A\in H} 2^{-|A|+1}$, and so $w_1 < 1$ because of the condition. That is the assumption of Maker's win led us to a contradiction.

Q. E. D.

Exercise 5.1 (Conway's frogs) *We consider the infinite chessboard (grid) with a line* l*, which will be the border of the "desert." There may be a finite number of stone (these are the "frogs") under the line* l*, such that each square may host at most one stone. If there are stones horizontally or vertically neighboring cells, and the the further one may jump on the empty cell, but then the closer one vanishes from the board. Show that*

- *There exists such a configuration, which may send up a frog (stone) up to 4 unit in to the desert.*
- *The high of 5 is unreachable for the frogs.*
- *One may modify the rules of jumping so that frogs placed diagonally can jump either. How far can they enter into the desert?*

Exercise 5.2 (E-S sharp for n-uniform hypergraph) *An (X, H) hypergraph is **n-uniform**, if $|A| = n$ for all sets $A \in H$. For all number n give a (X, H) n-uniform hypergraph, such that*

$$\sum_{A \in H} 2^{-|A|+1} = \sum_{A \in H} 2^{-n+1} = 1,$$

and Maker wins the (X, H) game.

Exercise 5.3 (The Erdős-Selfridge thm for classical problems) *Show that Breaker wins the Maker-Breaker versions of*

- *the $R(N, n)$ game, if $n > 2\log_2 N$,*
- *the $W(N, n)$ game, if $n > 2\log_2 N$,*
- *and the $HJ(n, d)$ game, if $d \leq n/\log_2 n$.*

Exercise 5.4 (The Erdős-Selfridge thm for reverse hypergraph games) *Both party can have a tie in the reversed (X, H) game, if*

$$\sum_{A \in H} 2^{-|A|+1} < 1,$$

and this inequality is sharp.

Exercise 5.5 (Painter-Chooser game) *For a hypergraph (X, H)* **Painter** *and* **Chooser** *play the following game. In every steps Chooser selects two uncolored elements of X, which the Painter must color to blue and red, or red and blue. (He is not allowed to color these elements by one color.) Chooser wins if at some stage of the game a one-colored (red or blue) edge $A \in H$ is created, while Painter wins otherwise. Show that if*

$$\sum_{A \in H} 2^{-|A|+1} < 1,$$

then Painter wins, but if equality holds, then Chooser win is possible.

Exercise 5.6 (The tenure game) *There is a university department with faculty members of diffent position. The chairman C and the dean D play the following (little absurd) game. C divides his faculty into two groups, D have to promote all members of one group, while he may fire the others. The goal of C is to secure a tenure position of at least one of his faculty. (Of course D wishes to get rid of all.) Let us denote by a_k the number of faculty members who need k promotion to reach tenured position. Show that*

- *If $\sum_{k=1}^{\infty} a_k 2^{-k} < 1$, then D wins.*
- *The numbers $x_1 \geq x_2 \geq \ldots \geq x_r$ are the negative powers of 2, such that $\sum_{i=1}^{r} x_i = 1$. Then the set $\{x_i\}_{i=1}^{r}$ can be divided into two parts in such a way, that the sum of numbers in both parts exactly $\frac{1}{2}$.*
- *If $\sum_{k=1}^{\infty} a_k 2^{-k} \geq 1$, then T wins.*

Exercise 5.7 (Filling up lemma) *During the (X, H) hypergraph game Breaker can guarantee that if in the kth step $A_k(I) = i$ and $A_k(II) = 0$, then $i \leq 1 + \log_2 |H|$.*

6 Probabilistic method and derandomization

"Maybe from the madness some beautiful will grow."

H.G. Wells, War of the Worlds (Musical version)

Out of the plethora of results we concentrate on only two areas. These are the connection between random colorings and games defined on hypergraphs and, thresholds for random graphs and graph games.

Long before the Erős-Selfridge theorem, Erdős had the following result.

Theorem 6.1 (Erdős) *If for a hypergraph (X, H) the $\sum_{A \in H} 2^{-|A|+1} < 1$, then $\chi(H) \leq 2$.*

Proof. Let us color the elements of X independently, and randomly to red and blue with probabilities 1/2-1/2. (That is the so-called *event space* Ω is the set of all two-colorings, and for the probability of each element $\omega \in \Omega$ we have $\Pr(\omega) = 2^{-|X|}$.) Independecy and disjointness imply that for an $A \in H$ $\Pr(A \text{ is monochromatic}) = 2^{-|A|+1}$. Let I_A the indicator function of such colorings in Ω, for which A is monochromatic, that is an $\omega \in \Omega$

$$I_{\mathcal{A}} = \begin{cases} 1 & \text{if in } \omega \ A \text{ is monochromatic} \\ 0 & \text{otherwise} \end{cases}$$

If T stands for the number of monochromatic edges, then $T = \sum_{A \in H} I_{\mathcal{A}}$. Let us compute the expected value of random variable T.

$$E[T] = E[\sum_{A \in H} I_{\mathcal{A}}] = \sum_{A \in H} E[I_{\mathcal{A}}] = \sum_{A \in H} 2^{-|A|+1} < 1,$$

where the linearity of expectation, and the fact that $E[I_{\mathcal{A}}] = P(\mathcal{A})$ holds for every indicator function were used.

Since $E[T] < 1$, there exists a coloring $\omega \in \Omega$ for which $T(\omega) < 1$. On the other hand, $T(\omega)$ is a natural number, we have $T(\omega) = 0$. To spell this out: there is no monochromatic edge $A \in H$, if one colors X according to ω.

Q. E. D.

This method was later referred as *first moment method.* The main importance of Erdős and Selfridge idea, that it yields a polynomial *derandomization* of the first moment method.

Random graphs, $G_{n,p}$

The most comfortable modell for random graphs is the $G_{n,p}$. We fix n vertices, then between every pairs of vertices draw an edge independently of each other with probability p. The resulting space Ω is called $G_{n,p}$.

A set $\mathcal{P} \subset G_{n,p}$ is *monotone graph property* if $G_1 \in \mathcal{P}$ and $G_1 \subset G_2$ (G_1 is a subgraph of G_2) implies that $G_2 \in \mathcal{P}$. (E. g. $\mathcal{C}_n$, the set of connected graphs on n vertices, $\mathcal{T}_n$, contaning a triangle, or $\mathcal{H}_n$ having Hamiltonian circuit.)

Assume that the empty graph on n vertices, $E_n \notin \mathcal{P}$, but $K_n \in \mathcal{P}$. Then the

$$f(p) := P(\{G \in G_{n,p} : G \in \mathcal{P}\})$$

function is monotone in p.

Theorem 6.2 (Erdős-Rényi) *If $p < (1-\epsilon)\log n/n$, then*

$$\lim_{n\to\infty} P(\{G \in G_{n,p} : G \in \mathcal{C}_n\}) = 0,$$

if $p > (1+\epsilon)\log n/n$, then

$$\lim_{n\to\infty} P(\{G \in G_{n,p} : G \in \mathcal{C}_n\}) = 1,$$

where $\epsilon > 0$.

Theorem 6.2 implies, if Maker and Breaker randomly take 1 and b edges of K_n, respectively, then for $b > n/((1-\epsilon)\log n)$ Maker almost surely get a spanning tree, while for $b < n((1+\epsilon)\log n)$ Breaker almost surely prevents this. These are surprisingly essentially the same bounds ($(1+\epsilon)n/\log n$ and $(\log 2 - \epsilon)n/\log n$, respectively) one obtains for assuming the perfect play of Maker and Breaker. Beware though, generally the random play may give different bounds!

Exercise 6.1 (Random colorings) *With the use of random colorings, show that*

- $\chi(R(N,n)) \leq 2$, *if* $2\log_2 N \leq n$.
- $\chi(W(N,n)) \leq 2$, *if* $2\log_2 N \leq n$.

Exercise 6.2 (Hamiltonian paths in a tournament) *T_n is a tournament on n vertices. A* **directed Hamiltonian path=DHP** *is such a directed path that contains all the vertices of T_n. Show that*

- *There is at least* 1 DHP *in every tournament T_n.*
- *The number of* DHP*'s is odd.*
- *There is a tournament T_n that contains at least $n!/2^{n-1}$* DHP*'s.*

Exercise 6.3 (Independence number) *Show that for every graph G on n vertices the followings hold for the independence number $\alpha(G)$*

- $\alpha(G) \geq \sum_{i=1}^{n}(d_i+1)^{-1}$, *where d_i is the degree of the ith vertex.*
- $\alpha(G) \geq n/(t+1)$, *where $t = n^{-1}\sum_{i=1}^{n} d_i$, i. e. the* **average degree**.

Exercise 6.4 (Turán's theorem for hypergraphs) *A hypergraph (X,H) is* **r-uniform**, *if $|A| = r$ for all set $A \in H$. A set $S \subset X$* **independent**, *if there is no such an edge $A \in H$ for which $A \subset S$. The* **independence number** *$\alpha(H)$ of (X,H) is the size of the biggest independent set of X. For a r-uniform hypergraph (X,H)*

- $\alpha(H) \geq np - |H|p^r$, *where $p \in (0,1)$ arbitrary.*
- *If $t = \frac{r|H|}{n}$, that is the* **average degree**, *then $\alpha(H) \geq (1-\frac{1}{r})\frac{n}{t^{\frac{1}{r-1}}}$.*

Exercise 6.5 (Covering sets) *For a hypergraph (X,H) the set $C \subset X$ halmaz is* **covering**, *if $C \cap A \neq \emptyset$ for all edge $A \in H$. The* **covering number** *$\tau(H)$ of the hypergraph (X,H) is the minimum size of a covering set. Show that*

- $\tau(H) \leq |X|p + \sum_{A\in H}(1-p)^{|A|}$, *for all $p \in (0,1)$*
- $\tau(H) \leq 2\sqrt{|X|}$, *if $A, B \in H$ and $A \neq B$ implies $|A| \neq |B|$.*
- *If there are no two circuits of the same lengths in G, which is a graph on n vertices, then the maximal number of edges in G is between $n+\sqrt{2n}$ and $n+2\sqrt{n+3\sqrt{n}}$.*

Exercise 6.6 (Markov's inequality) *For an integrable random variable X we have*

- *If $X \geq 0$, then $P(X \geq a) \leq E[X]/a$ for all $a > 0$.*
- *If for a constant M, $M \geq X \geq 0$, then $P(X > a) \geq (E[X]-a)/(M-a)$ for all $a < M$.*

Exercise 6.7 (Chebyshev's inequality) *If there exist $E[X]$ and $E[X^2]$ for a random variable X, and $\lambda > 0$ arbitrary, then*

$$P(|X - E[X]| > \lambda) \leq \frac{E[X^2] - E[X]^2}{\lambda^2}.$$

Exercise 6.8 (Lovász Local Lemma) *Assume that $A_1, ..., A_n$ are the events of a probability space $(\Omega, \mathcal{A}, P)$. The graph G is the* **dependence graph** *of $A_1, ..., A_n$, if $V(G) = \{1, ..., n\}$ and for all i A_i is independent of the σ-algebra generated by the events az $\{A_j\}_{j\in J}$, where $J = V(G) \setminus N(i)$, that is the vertices not incident to i. Show that if one has $P(A_i) \leq p$ for $A_1, ..., A_n$, $deg(i) \leq d$ for all i in G, and $4dp < 1$, then $P(\cap_{i=1}^{n}\overline{A}_i) > 0$. (Hint: Use induction on s. Show that if $S \subset \{1, ..., n\}$ and $|S| \leq s$, then $P(A_i | \cap_{j\in S}\overline{A}_j) \leq 2p$ for all i.)*

Exercise 6.9 (Lovász Lemma for hypergraphs) *Let $deg_H(x) = |\{A : A \in H,\ x \in A\}| < 2^{k-3}/k$ for all $x \in X$ of a k-uniform hypergraph (X, H). Then $\chi(H) \leq 2$.*

Exercise 6.10 (van der Waerden lower bound) *Let W be the hypergraph of the $W(N, n)$ (the van der Waerden) game. Then $\chi(W) \leq 2$, provided that $N \leq 2^n/8n$.*

7 Weight functions used by Maker

"The victory is the main goal of the war. In a prolonged war the weapons go blunt and the morale decreases."

Sun-tzu, The Art of War

The question is, if Maker can profit the use of weight funtions in a Maker-Breaker game?

Theorem 7.1 (Beck) *Let us suppose that in a Maker-Breaker game for an n-uniform hypergraph (X, H) it holds that any two element of X are at most in d edges at the same time. If $|H| > d2^{n-3}|X|$, then Maker wins.*

Proof. As before $x_1, x_2, ..., x_i$ and $y_1, y_2, ..., y_i$ are the elements of X taken by Maker and Breaker, respectively. The weight of an edge $A \in H$ *after* the i pair of steps, (i. e. after selection of y_i) is

$$w_i(A) = \begin{cases} 2^{-|A|+A_i(I)} & \text{if } A(II) = 0 \\ 0 & \text{otherwise} \end{cases},$$

where $A_i(I) = |A \cap \{x_1, ..., x_i\}|$ and $A_i(II) = |A \cap \{y_1, ..., y_i\}|$. As before the weight of $x \in X$ is $w_i(x) = \sum_{x \in A} w_i(A)$, and the total weight is $w_i = \sum_{A \in H} w_i(A)$. Maker follows the greedy algorithm, that is selects x_{i+1} such that $w_i(x_{i+1}) = \max\{w_i(z) : z \in X \setminus \{\{x_1, ..., x_i\} \cup \{y_1, ..., y_i\}\}\}$. Then $w_{i+1} \geq w_i - d/4$ holds. In fact, the selection of x_{i+1} doubles the value of sets containing x_{i+1}, while the answer of Breaker (y_{i+1}) decreases the total weight by such an amount that is the sum of *new* weights for the sets containing y_{i+1}. (Note, that y_{i+1} makes the value of the sets zero, which values just have been doubled by x_{i+1}.) Formally,

$$w_{i+1} = w_i + w_i(x_{i+1}) - w_i(y_{i+1}) - \psi_i,$$

where

$$\psi_i = \sum_{A \in H, \{x_{i+1}, y_{i+1}\} \subset A} w_i(A).$$

Since $w_i(A) \leq 1/4$, if $\{x_{i+1}, y_{i+1}\} \subset A$, so $\psi_i \leq d/4$. Furthermore $w_i(x_{i+1}) \geq w_i(y_{i+1})$, which implies

$$w_{i+1} = w_i + w_i(x_{i+1}) - w_i(y_{i+1}) - d/4.$$

Now induction shows that

$$w_i \geq w_{i-1} - d/4 \geq w_{i-2} - 2d/4 \geq ... \geq w_0 - id/4.$$

On the other hand $w_0 = \sum_{A \in H} 2^{-|A|} = 2^{-n}|H|$ and $i \leq |X|/2$, that is

$$w_{|X|/2} \geq 2^{-n}|H| - \frac{d|X|}{8} > 0$$

because of the conditions of the theorem. It means, that at the end of the game there must be an edge $A \in H$, such that $w_i(A) > 0$. But it can be only, if Maker has taken all the elements of A.

Q. E. D.

Exercise 7.1 (Maker-Breaker van der Waerden games) *Maker wins the Maker-Breaker $W(N,n)$ game, if $N \geq n^3 2^n$.*

Exercise 7.2 (Maker-Breaker Ramsey games) *Maker wins the Maker-Breaker $W(N,n)$ game, if $N \geq n^{k+3} 2^{\binom{n}{k}}$.*

Exercise 7.3 (Maker-Breaker Hales-Jewett games) *Maker wins the Maker-Breaker $W(N,n)$ game, if $d > \frac{\log 2}{2} n^2$.*

Exercise 7.4 (Induced subgraph game) *For a fixed graph G, and N natural number we define the $Sp(G,N)$ game. Maker and Breaker color the edges of K_N, by taking turns Maker wins, if there is going to be such a $|V(G)|$ element subset of vertices in K_N in which the edges colored by Maker isomorphic to G, while Breaker has colored all other edges. Let C_5 be the diagonal-free circuit of 5-length. Show that Maker wins the $Sp(C_5,N)$ game, if $N > 3842$.*

Exercise 7.5 (Colored van der Waerden games) *Maker and Breaker alternately colors the set $\{1,...,N\}$ to blue and red, respectively. Let us fix a function $f : \{1,...,n\} \mapsto \{blue, red\}$, and say that Maker wins, if they create such an n-lenght aritmethic progression $\{a_i\}_{i=1}^n \subset \{1,...,N\}$, for which the color of a_i is $f(i)$. Show that Maker wins, if $N \geq n^3 2^n$.*

8 The degree game

This game seemingly is not hypergraph game; it is though, just it is more enlightening to put in a different form.

Maker and Breaker alternately color the squares of an $n \times n$ board by their on color. Maker goal is to create such a column or row, in which a number of squares colored by him is at least $n/2 + k$. (Originally it was defined by the coloring the edges of $K_{n,n}$, that is why the name.)

Theorem 8.1 (Székely-Beck) *The following holds for the degree game, if n is big enough*
(i) if $k \geq \sqrt{n}\sqrt{\log n}$, then Breaker wins,
(ii) if $k \leq \frac{1}{12}\sqrt{n}$, then Maker wins.

Proof. (i) Let X be the squares of the board, and the columns and rows be edges in the hypergraph (X, H). Maker's and Breaker steps up to the ith turn are $x_1, ..., x_i$ and $y_1, ..., y_i$ respectively. Let us consider the weight function for an edge (column or row) $A \in H$:

$$w_i(A) = (1+\lambda)^{|\{x_1,...,x_i\} \cap A| - (n/2+k)} (1-\lambda)^{|\{y_1,...,y_i\} \cap A| - (n/2-k)},$$

where $\lambda > 0$ will be specified later. As usual, the weight of an $x \in X$ elem is

$$w_i(x) = \sum_{x \in A} w_i(A),$$

while the total weight is

$$T_i = \sum_{A \in H} w_i(A).$$

Let y_i be of maximal weight among the choosable elements. This weight function significantly differs from the one use in the proofs before, since the weight of an edge $A \in H$ do not vanishes when Breaker takes one of its element. We still have two inequalities

$$T_{i+1} \leq T_i - \lambda w_i(y_i) + \lambda w_i(x_{i+1}) \leq T_i,$$

$$T_1 \leq 2n(1+\lambda)(1+\lambda)^{-n/2+k}(1-\lambda)^{-(n/2-k)} < 4n(1-\lambda^2)^{-n/2}(\frac{1-\lambda}{1+\lambda})^k.$$

Therefore

$$T_i < 4n(1-\lambda^2)^{-n/2}(\frac{1-\lambda}{1+\lambda})^k$$

for all $i = 1, ..., n^2/2$. Let us suppose on contrary that Maker managed to color $n/2 + k$ squares in edge, say in the ith step. Of course it means

$$1 \leq T_i < 4n(1-\lambda^2)^{-n/2}(\frac{1-\lambda}{1+\lambda})^k,$$

or which is equivalent to this

$$(1-\lambda^2)^{n/2}(\frac{1+\lambda}{1-\lambda})^k < 4n.$$

Let us fix now $k = \sqrt{n \log n}$ and $\lambda = \sqrt{\frac{\log n}{n}}$. Using the approximations $1 - \epsilon \approx e^{-\epsilon}$ and $1 + \epsilon \approx e^{\epsilon}$ ($\epsilon > 0$ but small enough) we get

$$(1-\lambda^2)^{n/2} \approx e^{-\lambda^2 n/2} = e^{-\frac{\log n}{2}} = \frac{1}{\sqrt{n}},$$

$$(\frac{1+\lambda}{1-\lambda})^k = (1 + \frac{2\lambda}{1-\lambda})^k > (1+2\lambda)^k \approx e^{2\lambda k} = e^{2\log n} = n^2.$$

This implies

$$4n > (1-\lambda^2)^{n/2}(\frac{1+\lambda}{1-\lambda})^k \approx \frac{1}{\sqrt{n}} n^2 = n^{3/2},$$

which leads to a contradiction, if n is big enough, so we are at home.

Q. E. D.

The proof of (ii) is more technical, we will give motivation, a "wrong proof", and some hints how to correct this idea.

According to the random heuristic, we may expect Maker's win for $k = c\sqrt{n}$, where c is a constant. Indeed, playing random the number of blue squares would be the random variable $X_1 + X_2 + ... + X_n$, where the X_i are i.i.d.'s and $P(X_1 = 1) = P(X_1 = 0) = 1/2$. This follows the binomial distribution with expected value $E(\sum_{i=1}^n X_i) = n/2$, and variance

$$Var(\sum_{i=1}^n X_i) = E(\sum_{i=1}^n X_i - E(\sum_{i=1}^n X_i))^2 = \sum_{i=1}^n E(X_i - \frac{1}{2})^2 = n/4.$$

That is the standard deviation $\sigma(\sum_{i=1}^n X_i) = \sqrt{n}/2$, which means about such a deviation from $n/2$ almost surely. We may say the method which we outline is nothing, but the *derandomization of second moment method.*

Let the weight of an edge A be

$$w_i(A) = A_i(I) - A_i(II),$$

where $A_i(I) = |A \cap \{x_1, ..., x_i\}|$ and $A_i(II) = |A \cap \{y_1, ..., y_i\}|$. For an $x \in X$ the weight is

$$w_i(x) = \sum_{x \in A} w_i(A),$$

and the total weight is

$$T_i = \sum_{A \in H} w_i^2(A).$$

Maker colors the square x_{i+1} in the $(i+1)$st step, for which the value $w_i(x_{i+1})$ maximal. This effects T_i as follows. If $x_{i+1} \in A$ and $y_{i+1} \notin A$, then

$$w_{i+1}^2(A) = (A_i(I) - A_i(II) + 1)^2 = (A_i(I) - A_i(II))^2 + 2(A_i(I) - A_i(II)) + 1 =$$

$$= w_i^2(A) + 2w_i(A) + 1.$$

If $x_{i+1} \notin A$ and $y_{i+1} \in A$, then

$$w_{i+1}^2(A) = (A_i(I) - A_i(II) - 1)^2 = (A_i(I) - A_i(II))^2 - 2(A_i(I) - A_i(II)) + 1 =$$

$$= w_i^2(A) - 2w_i(A) + 1.$$

Let us suppose that $x_{i+1} \in A \cap B$ és $y_{i+1} \in C \cap D$, where A, B, C, D rows and columns of the board. Since $w_i(x_{i+1})$ were maximal,

$$w_i(A) + w_i(B) \geq w_i(C) + w_i(D).$$

That is if A, B, C and D are all different, then

$$T_{i+1} \geq T_i + 4,$$

while if for example $A = C$, then

$$T_{i+1} \geq T_i + 2,$$

that is the total weight T_i increases at least by 2 in every step. Since the total game consists of $n^2/2$ steps, we have

$$T_{n/2} \geq T_0 + 2n^2/2 = n^2.$$

On the other hand

$$n^2 \leq T_{n^2/2} = \sum_{A \in H} w_i^2(A) \leq 2n \max\{w_{n^2/2}^2(A) : A \in H\},$$

since $|H| = 2n$. Therefore there exists such an row or column A^*, for which

$$w_{n^2/2}^2(A^*) \geq n/2 \;\text{ i. e. }\; |w_{n^2/2}^2(A^*)| \geq \sqrt{n/2}.$$

This mean that *some* of the players colored at least $\sqrt{n/2}$ more squares of A^* than his opponent. Unfortunatelly, it might have been Breaker! The outline of the fix is bellow.

Exercise 8.1 (The degree game) *Define a new weight function*

$$w_i(A) = |A_i(I) - A_i(II) + \frac{\sqrt{n}}{4}|^+,$$

where

$$|\alpha|^+ = \begin{cases} \alpha & \text{if } \alpha > 0 \\ 0 & \text{if } \alpha \leq 0 \end{cases}.$$

For an $x \in X$, the weight $w_i(x)$ and the total weight T_i are definied as before. During the game a in the $i+1$st step the situation is **stagnating**, *if $x_{i+1}, y_{i+1} \in B$ and $w_i(B) = 0$ for a $B \in H$. (That is B contains at least $\sqrt{n}/4$ more squares of Breaker than that of Maker.) Prove the statements bellow.*

- *If the $i+1$st step is not stagnating, then $T_{i+1} \geq T_i + 2$, otherwise $T_{i+1} \geq T_i$.*
- *If the ith step is stagnating, then $w_i(C) = 0$, where C is the* **other** *edge, containing y_{i+1}. Furthermore if z is an uncolored element of A-nak, and D is an edge containing z-t, then $w_i(D) = 0$.*

Now we consider two cases. In the first we assume that out of the $n^2/2$ steps, less than $n^2/4$ (the half) were stagnating.

- $$T_{n^2/2} \geq n^2/2 \quad \max\{w_{n^2/2}^2(A) : A \in H\} \geq n/4,$$

  and there exists such an edge $A^* \in H$, in which Maker has colored at least $\sqrt{n}/4$ more elements than Breaker.

So we may assume more than half of the step were stagnating. We may also assume that in at least $n^2/8$ stagnating steps the squares x_{i+1} and y_{i+1} are in some, say row. Furthermore, if the $i+1$st step is stagnating, let $STAG(A,i)$ be the set of such squares of the row $A \ni x_{i+1}, y_{i+1}$ which are uncolored after the ist step.

- *There exists an i_0, for which*
$$|STAG(A,i_0)| \geq \frac{n}{4}.$$
- *After every steps*
$$\sum_{D \ column} (D(I) - D(II)) = 0.$$
Let
$$\sum_1 = \sum_{\substack{D \ column \\ D(I)-D(II)>0}} D(I) - D(II),$$
$$\sum_2 = \sum_{\substack{D \ column \\ D(II)-D(I)>0}} D(II) - D(I).$$
-
$$\sum_1 = \sum_2$$
- *After the step i_0th,*
$$\sum_2 \geq \frac{n^{3/2}}{16} \ \text{ and so } \ \sum_1 \geq \frac{n^{3/2}}{16}.$$
- *In the step i_0th*
$$\frac{3}{4} n \max\{D(I) - D(II) : D \ \ column\} \geq \sum_1,$$
that is
$$\max\{D(I) - D(II) : D \ \ column\} \geq \frac{\sqrt{n}}{12},$$
which concludes the theorem.

9 Biased games and infinite hypergraphs

"As no better man advances to take this matter in hand, I hereupon offer my own poor endeavor. I promise nothing complete; because any human being supposed to be complete, must for that very reason infallibly be faulty. I shall not pretend to a minute anatomical description of the various species, or - in this place at least - to much of any description. My object here is simply to project the draught of systematization of cetology. I am the architect, not the builder."

Herman Melville, Moby Dick

A (X, H, p, q) hypergraph game is *biased*, if $p \neq q$. First we generalize the Erdős-Selfridge theorem, then show an application. The main use of this games will show in the investigation of infinite games, since the natural strategy is to cut X into parts, and define *auxiliary* games on each part. (The matchings are just special case of this. The division is even more important when we use weight function, for that most of the weight would be infite otherwise.)

Theorem 9.1 (Beck) *Breaker wins the Maker-Breaker version of the* $(X, H, p, 1)$ *hypergraph game, if*

$$\sum_{A \in H} 2^{-|A|/p} < \frac{1}{2}.$$

Proof. We do almost the same as before. The ith steps of Maker are $x_i^1, ..., x_i^p$, while Breaker steps is y_i. The weight of $A \in H$ before the ith step of Breaker is

$$w_i(A) = \begin{cases} \lambda^{A_i(I) - |A|} & \text{if } A_i(II) = 0 \\ 0 & \text{otherwise} \end{cases},$$

where $A_i(I) = |A \cap \{x_1^1, ..., x_1^p, ..., x_i^1, ..., x_i^p\}|$, $A_i(II) = |A \cap \{y_1, ..., y_{i-1}\}|$, és $\lambda = 2^{1/p}$. The weight of $x \in X$, and the total weight are $w_i(x) = \sum_{x \in A} w_i(A)$ and $w_i = \sum_{A \in H} w_i(A)$, respectively. Breaker follows the greedy algorithm.

When Maker takes an element of the edge A, the weight of A becomes multipied by λ. In the worst case all of Maker's steps are in A, so the weight can be as high as $w_i(A)\lambda^p$. That is the increase is $(\lambda^p - 1)w_i(A)$. Now

$$w_i - w_i(y_i) + (\lambda^p - 1) \max_{1 \leq j \leq p} (w_i(x_i^j)) \geq w_{i+1},$$

from which the maximality of $w_i(y_i)$, and $\lambda^p = 2$ implies that $w_i \geq w_{i+1}$. It can be argued analogously that

$$w_1 \leq \sum_{A \in H} \lambda^{-|A| + p} = \sum_{A \in H} 2^{\frac{-|A|}{p} + 1} < 1.$$

On the other hand, if Maker would win, say in the kth step then would be an edge $A \in H$, for which $1 = w_k(A) \leq w_k \leq w_1 < 1$.

Q. E. D.

One may proof the most general case, when $q \neq 1$. Note that these are all sharp.

Theorem 9.2 (Beck) *Breaker wins the* (X, H, p, q) *hypergraph game if*

$$\sum_{A \in H} (1+q)^{-|A|/p} < \frac{1}{1+q}.$$

10 Spanning tree game

This is the Shannon switching game for $G = K_n$, but to balance Breaker huge disadvantage, he may select p, while the opponent just 1 edge in each turn. According to the random heuristic with appropriate constants c_1 and c_2 Breaker wins if $p > c_1 n/\log n$, while Maker wins if $p < c_2 n/\log n$. The summary of results of Chvátal, Erdős and József Beck is as follows.

Theorem 10.1 *In the spanning tree game I Breaker wins, if* $p > (1+\epsilon)n/\log n$, *but Maker wins if* $p < (\log 2 - \epsilon)n/\log n$, *where* $\epsilon > 0$, *and* $n > n_0(\epsilon)$.

We prove only the second part, as it being the straightforward consequence of Theorem 9.1.

Proof. The method of Theorem 7.1 breaks down, we need a new idea. It appeared in exercise 3.3 that a Maker-Breaker game (X, H, p, q) can be rewritten in a form of an (X, H^*, p, q) game, where (X, H^*) is the *dual hypergraph* of (X, H), that is

$$H^* = \{A : A \subset X \text{ minimal and } A \cap B \neq \emptyset,\ \forall B \in H\}.$$

It is well-known that if $H = \{F : F \text{ spanning tree in } K_n\}$, then $H^* = \{V : V \text{ is a cut of } K_n\}$ (see exercise 3.9).

If V is an (A, B)-cut, where $|A| = k$, $|B| = n - k$, then $|V| = k(n-k)$, and there are exactly $\binom{n}{k}$ cuts dividing the vertices of K_n in proportion $k : n - k$. So the condition of Theorem 9.1 reduces to

$$\sum_{k=1}^{k=\lfloor n/2 \rfloor} \binom{n}{k} 2^{-k(n-k)/p} < \frac{1}{2}.$$

If we plug in the inequality $p < (\log 2 - \epsilon)n/\log n$ into that sum, we get the following estimation

$$k = 1: \quad \binom{n}{1} 2^{-1(n-1)\frac{\log n}{n \log 2 - \epsilon}} \approx 2^{-\frac{\log n}{\log 2 - \epsilon}} = (e^{-\log n})^{1+\epsilon} = \frac{1}{n^{1+\epsilon}}$$

$$k = 2: \quad \binom{n}{2} 2^{-2(n-2)\frac{\log n}{n \log 2 - \epsilon}} \approx \left(\frac{1}{n^{1+\epsilon}}\right)^2 \binom{n}{2} \approx \left(\frac{1}{n^{\epsilon}}\right)^2$$

ad so on

$$k = i: \quad \binom{n}{i} 2^{-i(n-i)\frac{\log n}{n \log 2 - \epsilon}} \approx (\frac{1}{n^{1+\epsilon}})^i \binom{n}{i} \approx (\frac{1}{n^{\epsilon}})^i.$$

That is the sum can be bounded (asymptotically) by the geometric series

$$\sum_{k=1}^{\infty} (\frac{1}{n^{\epsilon}})^k,$$

in which the first element $\frac{1}{n^{\epsilon}} < 1/2$, if n is big enough, implying that the sum is smaller than 1.

Q. E. D.

Remark. One of the most beautiful result for biased games defined on K_n is that of Beck. This one says that it is not much harder for Maker to achieve a Hamiltonian circuit than a spanning tree. In the $\mathrm{Ham}(p, 1, n)$ game Maker takes 1 edge, while Breaker p edges in each turn. Maker's goal is to get a Hamiltonian circuit.

Theorem 10.2 (Beck) *Maker has a winning strategy for the* $\mathrm{Ham}(p, 1, n)$ *game if* $p < (\frac{\log 2}{27} - \epsilon)n/\log n$, *where* $\epsilon > 0$ *and* $n > n_0(\epsilon)$.

Exercise 10.1 (48-in-a-row) *Give a winning strategy to Breaker for the game k-in-a-row, which is based on weight functions.*

Exercise 10.2 (Biased k-in-a-row) *If Maker takes two, while Breaker only one square in each step, then Maker wins the k-in-a-row for every natural k.*

Exercise 10.3 (Harary's game) *Fix a polyomino* P, *that is a finite, connected set of squares of the infinite chessboard. The* $P(1, 1)$ *game is in which Maker wins by occupying all squares of a polyomino* P^* *on the board which is the copy of* P. *(Maker and Breaker take one square in every steps.) Show that Breaker wins if*

- P *is a rectangle of sizes* 5×1, *or*
- P *is a square of size* 2.

Exercise 10.4 (Triangle game) *Maker takes* 1 *while Breaker* $c\sqrt{n}$ *edges of* K_n *in every steps, where* c *is a constant. Maker wins by getting a* K_3 *(triangle). Show that*

- *Maker wins, if* $c \leq 1/100$,
- *Breaker wins, if* $c \geq 100$.

Exercise 10.5 (Recycled k-in-a-row) *Stop the k-in-a-row after the first* n *step, and then play with the stones put on the board like in Nine Man Morris (Mill). Show that Maker cannot win, if* $k \geq 9$. *What can we claim for* $k = 8$*?*

Exercise 10.6 (Accelerated k-in-a-row 1.) *The k-in-a-row may be played in such a way, that both player mark n squares in each step; this game is called $A_k(n, n)$ game. Prove that Breaker wins, if $k \geq 3(n + 3)$.*

Exercise 10.7 (Accelerated k-in-a-row 2.) *Prove that Maker wins the $A_k(n, n)$ game, if $k \leq n + \frac{\log_2 n}{\log_2 \log_2 n}$.*

Exercise 10.8 (Accelerated k-in-a-row 3.) *Prove that Breaker wins the game $A_k(n, n)$, if $k \geq n + 1000 \log n$, provided that $n \geq 1000$.*

Exercise 10.9 (General form of Theorem 7.1) *Maker wins the Maker-Breaker version of the $(X, H, 1, q)$ hypergraph game, if*

$$\sum_{A \in H} (1 + q)^{-|A|} > q^2 (1 + q)^{-3} d_2(H) |X|,$$

where

$$d_2(H) = \max\{|H_{x,y}| : H_{x,y} \subset H, A \in H_{x,y} \Longrightarrow x, y \in A\}.$$

Exercise 10.10 (Tournaments 2.) *Maker and Breaker play the game defined in exercise 4.6, but now T_n is arbitrary. (Recall that Maker wins by getting a copy of T_n in K_N.) Show that Maker wins, if N is big enough.*