Application of fuzzy set theory in image analysis

Analysis and defuzzification of discrete fuzzy spatial sets

Who am I?

- Nataša Sladoje

Teaching assistant in mathematics
Department of fundamental disciplines
Faculty of Engineering
University of Novi Sad

B.Sc. in Mathematics, Univ. of Novi Sad, Serbia
M.Sc. in Discrete mathematics, Univ. of Novi Sad, Serbia
Ph.D. in Image analysis, Centre for Image Analysis, Uppsala, Sweden

Questions to be answered today

- What is a fuzzy set?
- What is a fuzzy segmented image?
- What are the advantages of using them?
- How do we relate fuzzy segmented images to conventional binary images?
- What is defuzzification?
- How do we perform it?
- Any conclusions?

Questions to be answered today

- What is a fuzzy set?
- What is a fuzzy segmented image?
- What are the advantages of using them?
- How do we relate fuzzy segmented images to conventional binary images?
- What is defuzzification?
- How do we perform it?
- Any conclusions?

What is a fuzzy set?

Btw, what is a set? "... to be an element..."

Let us observe a (crisp) reference set
$X=\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$.

Let us form

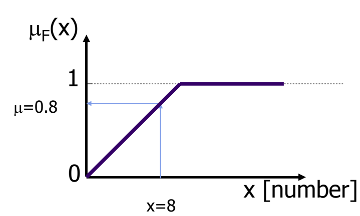
- A (crisp) subset C of X, $C = \{x \mid 3 < x < 8\}$.
  $C = \{4, 5, 6, 7\}$. (Easy! "Yes, or no" ...)
- A set F of big numbers in X (?!)
  $F = \{10, 9, 8, 7, 6, 5, 4, 3, 2, 1\}$

(Yes or no? ... More like graded ...)

What is a fuzzy set?

All the numbers in X belong to F
- all of them are big!! - but to different extents.

$\mu_F(8) = 0.8$



Fuzzification

The process of assigning to each element of a reference set its degree of belongingness to a fuzzy set.

To define a fuzzy set ↔ To define a membership function

A fuzzy set F of a reference set X is a set of ordered pairs

$$F = \{ (x, \mu_F(x)) \mid x \in X \}$$

where $\mu_F : X \to [0,1]$

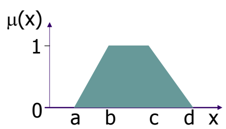
An example: Fuzzy set of *tall men*.

Name	Height, cm	Degree of Membership	
		Crisp	*Fuzzy*
Chris	208	1	1.00
Mark	205	1	1.00
John	198	1	0.98
Tom	181	1	0.82
David	179	0	0.78
Mike	172	0	0.24
Bob	167	0	0.15
Steven	158	0	0.06
Bill	155	0	0.01
Peter	152	0	0.00

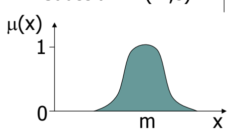
The elements of the fuzzy set "tall men" are all men, but their degrees of membership depend on their height

Types of Membership Functions

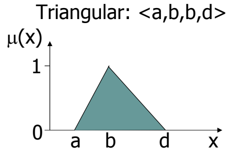
Trapezoid: <a,b,c,d>



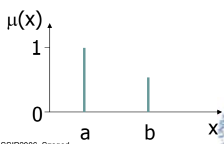
Gaussian: N(m,s)



Triangular: <a,b,b,d>



Singleton: (a,1) and (b,0.5)



What is a fuzzy set?

- Number 10 is not probably big!
- ...and number 2 is not probably not big.

Uncertainty is a concequence of non-sharp boundaries between the notions/objects, and not because of lack of information.

Terminology

An α-cut of a set A, for α ∈ (0,1], is a crisp set A_α

$$A_\alpha = \{x \in X \mid \mu_A(x) \geq \alpha\}.$$

The support of a fuzzy set A is a set of elements of *X*

$$Supp(A) = \{x \in X \mid \mu_A(x) > 0\}.$$

The core of a fuzzy set A is a set of elements of *X*

$$Core(A) = \{x \in X \mid \mu_A(x) = 1\}.$$

To "fuzzify" a function *f* :

(1) apply it to each α-cut

(2) integrate over α-cuts

Stacking of α-cuts

$$f(S) = \int_0^1 \hat{f}(S_\alpha)\, d\alpha \qquad f(S) = \frac{1}{\ell} \sum_{\alpha=1}^{\ell} \hat{f}(S_\alpha)$$

Examples:
Signature, area, perimeter

Questions to be answered today

- What is a fuzzy set?
- What is a fuzzy segmented image?
- What are the advantages of using them?
- How do we relate fuzzy segmented images to conventional binary images?
- What is defuzzification?
- How do we perform it?

Discrete spatial fuzzy sets

- Object of interest is represented as a (discrete) spatial fuzzy subset of a grid
- The mapping $\mu: X \rightarrow [0,1]$ becomes

$$\mu: Z \times Z \rightarrow \{0, 1, 2, \ldots, m\} \quad \text{(in 2D)}$$

m – maximal number of grey levels available (e.g., m=255 for 8-bit pixel representation)

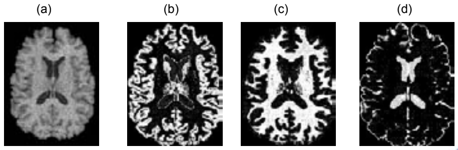
Spatial dimension(s) vs. fuzzy membership dimension

Fuzzy image is a grey-level image, but…

- (a) A sample slice from acquired MRI data set.
- Membership functions: (b) gray matter (GM), (c) white matter (WM), (d) cerebrospinal fluid (CSF).

(fuzzy c-means algorithm)

(a) (b) (c) (d)

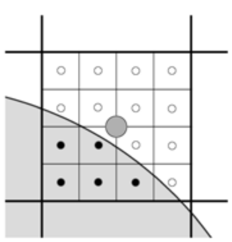


Objects with fuzzy borders

- Most of the pixels are easily classified as object pixels, or as background pixels
- Pixels close to the border of the object are more difficult to classify
- We assign to them a fuzzy membership value according to the extent of their belongingness to the object

area coverage approach

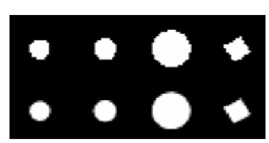
Area coverage approach



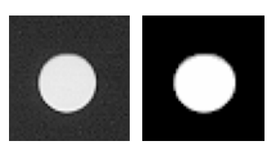
The discrete approximation of area coverage fuzzification facilitated by super-sampling of (border) pixels.

The membership of the presented pixel (divided into 16 sub-pixels) is 5/16.

Examples of objects with fuzzy borders



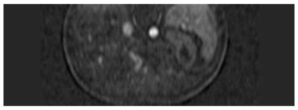
Synthetic objects. Discrete approximation of the area coverage approach is used to create objects with fuzzy borders.



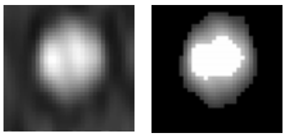
Left: Image of a circular hole in a piece of dark paper, obtained by a scanner set.

Right: Corresponding fuzzy segmentation.

Examples of objects with fuzzy borders



A cross-section of an MRA image of a human body. The bright circular object is the aorta.



Left: Aorta extracted.

Right: Fuzzy segmented image.

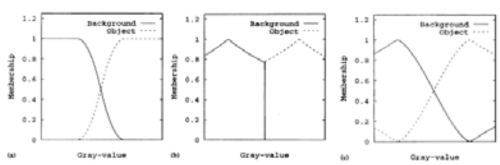
Fuzzy segmentation methods

Region-based methods

- Fuzzy thresholding
- Segmentation based on clustering
- Region-growing
- Object as a fuzzy connected component

etc.

Fuzzy thresholding



Membership distributions assigned using

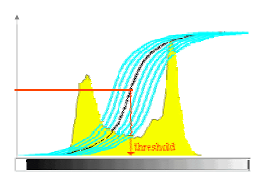
(a) Murthy and Pal (1990)

(b) Huang and Wang (1995)

(c) Fuzzy c-means (Bezdek 1981) algorithms.

Fuzzy thresholding by minimizing fuzziness in the image

Murthy and Pal's algorithm



S-function is applied to the grey-levels of the image.

A family of S-functions are tested, with different choices of cross-over points, and same band-width (interval with memberships within (0,1)).

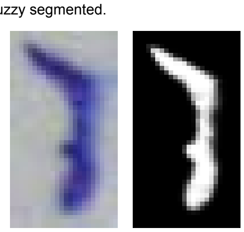
The S-function with parameters providing the minimal (index of) fuzziness in the image is selected.

The fuzzy segmented image is obtained by applying the selected S-function to the image.

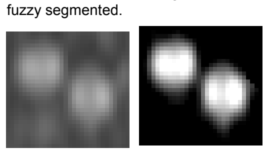
Thresholding at 0.5 provides crisp segmentation.

Fuzzy thresholding by minimizing fuzziness

A selected part of a microscopic image of a bone implant. A bone region is fuzzy segmented.



A slice of a 3D MRA image of a human aorta. A vessel region is fuzzy segmented.



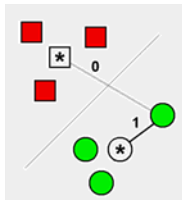
Fuzzy thresholding

Huang and Wang's algorithm

- Object and background are fuzzy sets with mutually exclusive supports;
- For each tested threshold, a point is assigned either to the object, or to the background, with the membership between 0.5 and 1;
- The closer the intensity of the point to the mean of the region, the higher its membership to that region;
- The threshold is chosen so that the entropy in the image is minimized.

Thresholding by using fuzzy c-means clustering algorithm

- Two clusters
- Partition grey levels into classes



Algorithm

-make initial guess for cluster means
-iteratively
    use the estimated means to assign
        samples to clusters
    update means
until there are no changes in means

Fuzzy c-means clustering

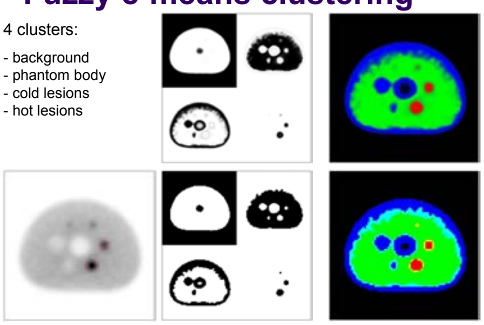
Bezdek

- a partition of the observed set is represented by a $c{\times}n$ matrix $U=[u_{ik}]$, where u_{ik} corresponds to the membership value (anything between 0 and 1!) of the k^{th} element (out of n), to the i^{th} cluster (out of c)
- boundaries between subgroups are not crisp
- each element may belong to more than one cluster – its ”overall” membership equals one
- objective function includes parameter controlling degree of fuzziness

Fuzzy c-means clustering

4 clusters:

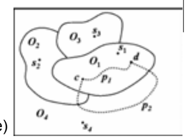
- background
- phantom body
- cold lesions
- hot lesions



Regions as fuzzy sets

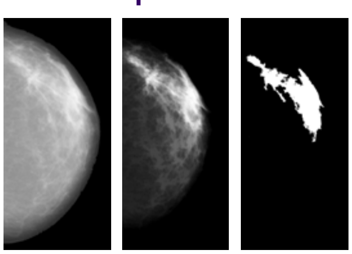
Regions as crisp sets

Object as a fuzzy connected component



- fuzzy connectedness combines
    - fuzzy adjacency (close in space)
    - fuzzy affinity (close in terms of intensities)
- strength of connectedness is assigned to each pair of points
- the weakest link of the strongest path determines the strength of a path
- the strength of a path determines the strength of connectedness between two points
- object – a fuzzy connected component of a given strength

An example



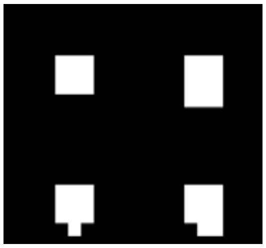
Breast density as measured from the volume of dense tissue in the breast is considered to indicate a risk factor for breast cancer.

A digitized X-ray mammogram, the fuzzy connectivity scene of a dense (fibroglandular) region (as opposed to fatty regions), and the segmented crisp region.

Questions to be answered today

- What is a fuzzy set?
- What is a fuzzy segmented image?
- What are the advantages of using fuzzy segmented images?
- How do we relate fuzzy segmented images to conventional binary images?
- What is defuzzification?
- How do we perform it?

Discrete disks (?!)



Descrete **crisp** representations of a continuous disk.

Gauss digitization is applied to the continuous shape (disk) whose centre position is moved within a pixel.

Advantages

- Expressing intrinsic fuzziness in images
- Information preservation
- Handling blurring, noise and background variation in a more robust way than crisp approaches
- Shape descriptors achieve much higher precision
- Fuzzy reasoning provides tools for improved image interpretation and understanding

Measurements

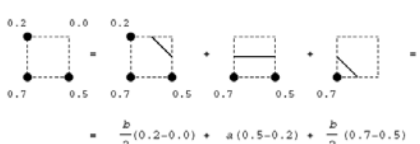
- Shape analysis often assumes performing various measurements of the shape properties
- Based on discrete shape representation, we estimate features of a real continuous imaged shape
- Estimation of area, perimeter, compactness, moments of higher order, signature of a shape all exhibit higher precision if estimated from the fuzzy, instead of the crisp representation of a shape

Example: area and perimeter of a fuzzy set

- Area (cardinality) – sum the membership values of all the points
- Perimeter – take into account the length of a border line **and** its "height"

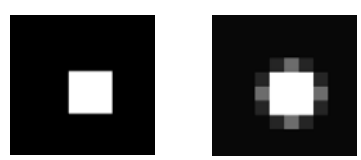
Recall: $f(S) = \frac{1}{\ell} \sum_{\alpha=1}^{\ell} \hat{f}(S_\alpha)$

How to calculate perimeter of a fuzzy set?



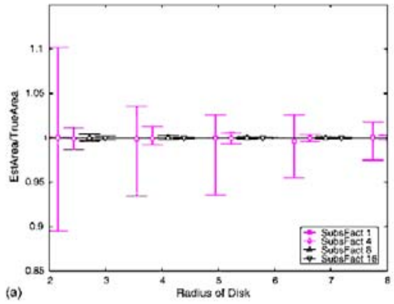
An example: Calculation of a contrubution to a perimeter for a Marching Square in a fuzzy object.

Features estimation

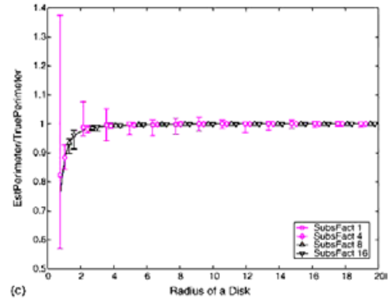


object	Radius	Perim	Area
real disk	1.9	11.94	11.34
crisp disk	–	10.27	12
fuzzy disk	–	11.6	11.31

Statistical study: precision of area estimation



Statistical study: precision of perimeter estimation



Geometric moments

- Geometric (*p,q*)-moment of an image *f(x,y)* is defined by

$$m_{p,q} = \sum_i \sum_j f(i,j) i^p j^q$$

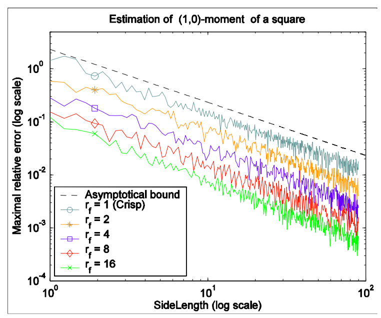
where (*i,j*) are coordinates of the integer sampling grid.

$m_{0,0}$ - area of the object

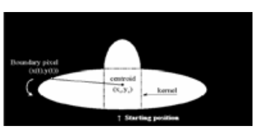
$\left(\frac{m_{1,0}}{m_{0,0}}, \frac{m_{0,1}}{m_{0,0}}\right)$ - centroid of the object

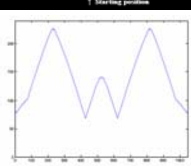
To increase spatial resolution of an image *r* times provides (asymptotically) the same precision as using r^2 membership values.

Precision of (1,0)-moment estimation of a convex shape



Shape signature based on distance from the centroid

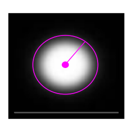




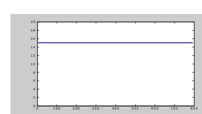
A star-shaped object with respect to the centroid and its corresponding signature.

- How to adjust it from continuous crisp to discrete fuzzy shape?
  - What is boundary?
  - Where is centre of gravity?
  - How to measure distance?

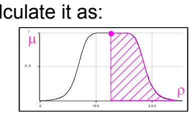
Signature of a disk (continuous case)



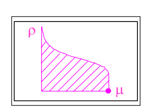
⇒



Calculate it as:



Radial integral of the membership function



Average of signatures of all α-cuts

Equivalent!!

Measuring distance in fuzzy sets

- length of a link

$$\max\{\mu(x),\mu(y)\}\times\|x-y\|$$

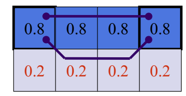
$$\frac{1}{2}(\mu(x)+\mu(y))\times\|x-y\|$$

- length of a path = sum of lengths of the links

What is shorter?

0.8 + 0.8 + 0.8 = 2.4

0.5 x 1.4 + 0.2 + 0.5 x 1.4 = 1.6

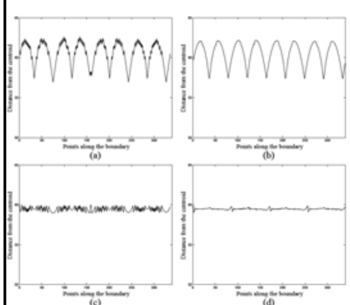


Shape signature

- Method 1 ("vertical")
  - Boundary of the support of a fuzzy set
  - Estimation of a length of digital straight line segments from centroid to the boundary points (radial summation of the memberships)
- Method 2 ("horizontal")
  - Averaged signatures obtained for each α-cut
  - Euclidean distance to the "thick" border

Note:The same in continuous, but not in discrete case!

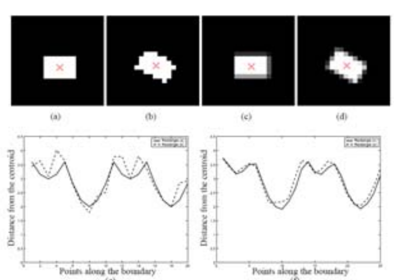
Example: signature of a disk



(a) Method 1 applied to a crisp disk
(b) Method 1 applied to a fuzzy disk

(c) Method 2 applied to a crisp disk
(d) Method 2 applied to a fuzzy disk

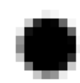
Signature: sensitivity to rotation



Conclusion: Fuzzy sets in image analysis

- Fuzzy sets provide a useful representation of image objects.
  - Simplifies the construction of stable and robust image processing methods.
  - Preservation of fuzziness implies preservation of information.

crisp

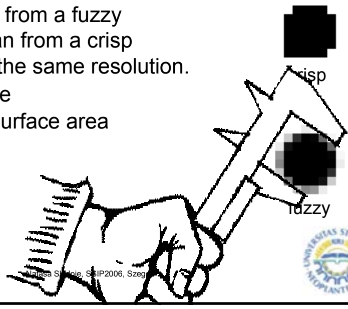


fuzzy

Conclusion: Fuzzy sets in image analysis

Many object features can be estimated with a higher precision from a fuzzy representation than from a crisp representation at the same resolution.

- Area/Volume
- Perimeter/Surface area
- Moments
- Signature
- …



Questions to be answered today

- What is a fuzzy set?
- What is a fuzzy segmented image?
- What are the advantages of using them?
- How do we relate fuzzy segmented images to conventional binary images?
- What is defuzzification?
- How do we perform it?
- Any conclusions?

Fuzzification & Defuzzification

Fuzziness is

- intrinsic quality of images
- outcome of most imaging devices

Consequently, two types of fuzziness

- inherent to the observed objects and images (preserve it!)
- imprecision and uncertainty due to noise (reduce it!)

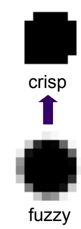
Sometimes crisp is required

Reasons for that may be:

- to provide easier and less subjective interpretation of images, especially if the dimension is higher than 2;
- in cases when fuzzy image analysis tools are not yet developed, while crisp analogues exist.

Defuzzification

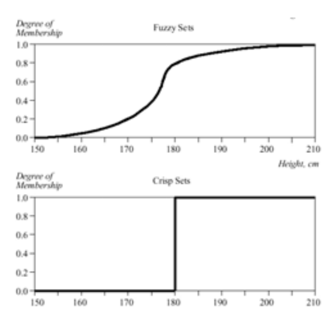
to generate a crisp representation of a fuzzy set.



Questions to be answered today

- What is a fuzzy set?
- What is a fuzzy segmented image?
- What are the advantages of using them?
- How do we relate fuzzy segmented images to conventional binary images?
- What is defuzzification?
- How do we perform it?
- Any conclusions?

An example: Fuzzy set of tall men

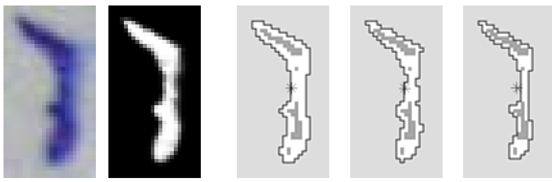


Fuzzy membership function

Defuzzification (by α-cut at α=0.8)

An example: Defuzzification by *α*-cutting

α-cutting is most common way to perform defuzzification to a set.



A part of a microscopy image of a bone implant. Original and a fuzzy segmented bone region. *α*-cut at *α*=0.25 *α*-cut at *α*=0.5 *α*-cut at *α*=0.75

Which *α* to choose?

Questions to be answered today

- What is a fuzzy set?
- What is a fuzzy segmented image?
- What are the advantages of using them?
- How do we relate fuzzy segmented images to conventional binary images?
- What is defuzzification?
- How do we perform it?
- Any conclusions?

Defuzzification to a (crisp) set

- Two views of defuzzification:
  - Mapping from a set of fuzzy sets to a set of crisp sets
  - Inverse mapping of fuzzification
- Defuzzification by α-cutting

  (thresholding of a membership function)
- Defuzzification by distance minimization

Defuzzification by distance minimization

Defuzzification formulated as an optimization problem

Definition: An optimal defuzzification of a fuzzy set A is a crisp set C (on the same reference set X) such that the distance between A and C is minimal:

$$\mathcal{D}(A) \in \{C \in \mathcal{P}(X) \mid d(A,C) = \min_{B \in \mathcal{P}(X)} [d(A,B)]\} .$$

Find the crisp set closest to the given fuzzy set.

Distance between fuzzy sets

The most common:

Minkowski distance based on the point-wise difference in membership values

$$d_p(A,B) = \left(\sum_{i=1}^{n} |\mu_A(x_i) - \mu_B(x_i)|^p\right)^{1/p}, \quad p \geq 1,$$

$$d_\infty(A,B) = \max_{i=1\ldots n} (|\mu_A(x_i) - \mu_B(x_i)|) .$$

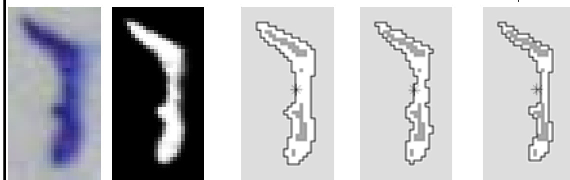
A crisp set B at minimal Minkowski distance (for any p≥1) from a fuzzy set A is the α-cut of A for α=0.5.

Tall men, one more time

Height, cm	Degree of Membership	
	Crisp	Fuzzy
208	1	1.00
205	1	1.00
198	1	0.98
181	1	0.82
179	0	0.78
172	0	0.24
167	0	0.15
158	0	0.06
155	0	0.01
152	0	0.00

$$\begin{aligned} d_1(F,C) &= |1-1.00| + |1-1.00| + |1-0.98| + |1-0.82| + \\ &\quad |0-0.78| + |0-0.24| + |0-0.15| + |0-0.06| + |0-0.01| + |0-0.00| \\ &= \mathbf{2.28} \\ d_\infty(F,C) &= \max(|1-1.00|, |1-1.00|, |1-0.98|, |1-0.82|, \\ &\quad |0-0.78|, |0-0.24|, |0-0.15|, |0-0.06|, |0-0.01|, |0-0.00|) \\ &= \mathbf{0.78} \end{aligned}$$

Defuzzification by minimizing point-wise difference



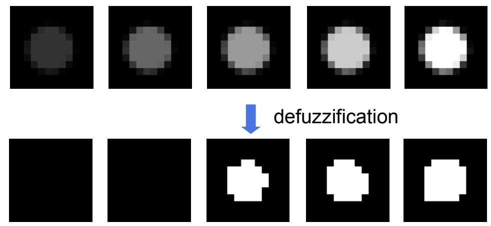
Original | Fuzzy set A | $A_{0.25}$ | $A_{0.5}$ | $A_{0.75}$

$d_1(A,A_{0.25})=52.8$ $d_1(A,A_{0.5})=45.2$ $d_1(A,A_{0.75})=55.7$

The α-cut at α=0.5 is an optimal defuzzification for this type of point-wise distance measures (for any value of p).

Defuzzification by minimizing point-wise difference

μ=0.2 μ=0.4 μ=0.6 μ=0.8 μ=1



Stability?

Preservation of shape

- To perform defuzzification based only on point-wise membership values puts too strong emphasis on the local structure.
- Inclusion of relevant global shape features into the distance measure ensures that the defuzzification resembles the fuzzy set with respect to these features.

Feature distance

- Extract both local and global "features" of the sets.
- Make a vector representation of each set using the extracted features.
- Measure the distance between two sets in this high dimensional feature space.

The feature distance between sets A and B, with feature representations $\Phi(A)$ and $\Phi(B)$, is

$$d^{\Phi}(A,B) = d(\Phi(A),\Phi(B))\,.$$

Decisions to make

- Which features to use in the defuzzification?
- What distance to use?
- How to optimize the distance?

- Features sometimes contradict each other, and often optimization for more features leads to worse matching of each of them.
- Minkowski distances for different values for p provide different results.
- Analytical solution of the problem cannot be found.
  It is needed to avoid local minima of the distance function.
  The search space is huge.

Which features to use?

We have studied the following features:

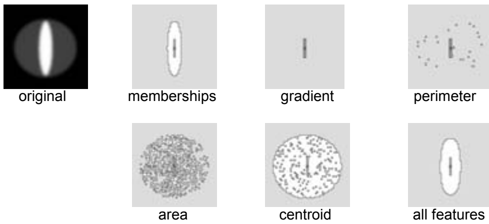
- local: membership values, gradient in each point,
- global: perimeter, area, and centroid (centre of gravity).

What distance measure to use?

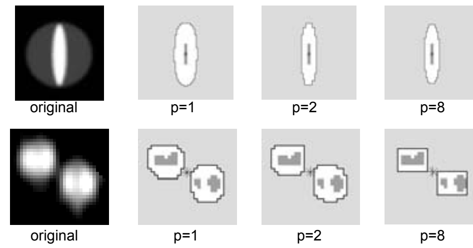
We use the Minkowski distance, p=1

$$d_p(\mathbf{x},\mathbf{y}) = \left(\sum_{i=1}^{n} |x_i - y_i|^p\right)^{1/p}$$

Examples of defuzzification based on different features



Examples of defuzzification by minimizing different distance measures



A fuzzy segmentation, and defuzzifications, of a slice in a 3D MRA of a human aorta, where it separates into the two illiac arteries.

Minimizing the distance

- An analytic solution is, in general, not know.
- 2^n configurations – exhaustive search is not an option.
- Some heuristics must be applied.

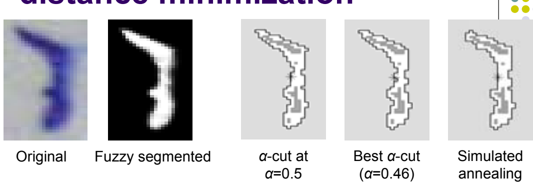
Simulated annealing has shown to provide a good trade off between speed and performance.

The optimal α-cut provides a fast alternative (and a good starting point for the simulated annealing).

Simulated annealing

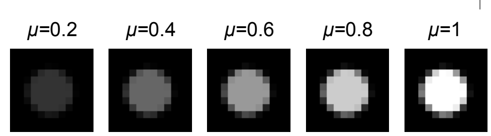
- Select one pixel in the reference set;
- If it is set, then unset; otherwise,set;
- Calculate the distance of the newly created crisp set to the fuzzy set;
- If better, then accept it;
- Otherwise, accept it with some probability that depends on the size of the change, and the status ("temperature") of the process;
- Select a new point, and repeat;
- Change parameters for making decision of accepting;
- Repeat until the termination criterion is fulfilled.

Defuzzification by feature distance minimization

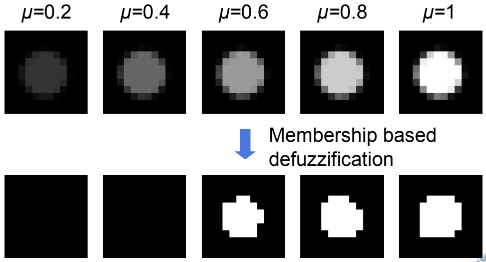


	Perimeter	Area	Centroid	Distance
Fuzzy object	91.46	182.78	(13.66,21.08)	0
α-cut at α=0.5	92.75	176.00	(13.57,21.18)	0.1271
α-cut at optimal α	93.30	182.00	(13.68,21.10)	0.1177
Sim. annealing	91.48	182.00	(13.65,21.08)	0.1053

Improved stability from global features

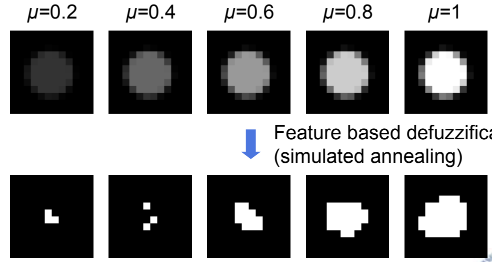


Improved stability from global features



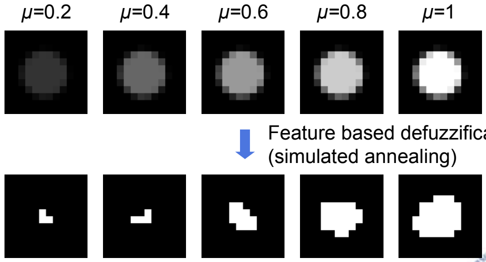
Stability?

Improved stability from global features



Better stability

Additional constraints in the defuzzification



With topological constraints

Even better: High resolution defuzzification

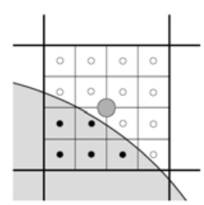
- Recall: Features of an original crisp continuous object can be estimated with a higher precision from a fuzzy representation, than from a crisp representation at the same spatial resolution.
- In other words: A fuzzy representation can compensate for lack of spatial resolution.
- We want to use that information to find a crisp representative at an increased spatial resolution.

Distance between sets at different resolution

De-couple the feature representation from the image resolution.

- Global features
  - By proper rescaling of the feature values.
- Local features
  - By comparing individual pixels of the lower resolution image, with blocks of pixels of the higher resolution image.

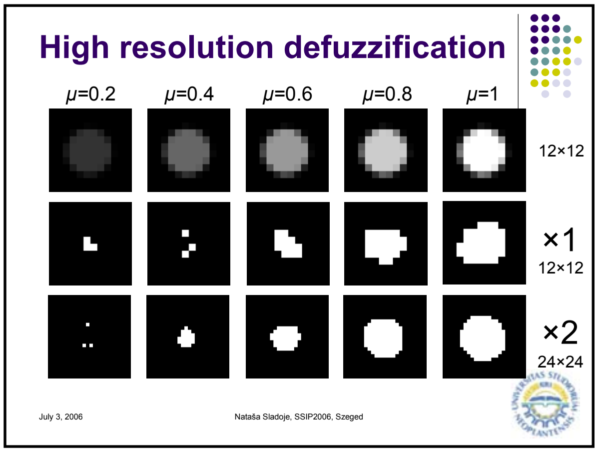
Membership value – area of a block of pixels

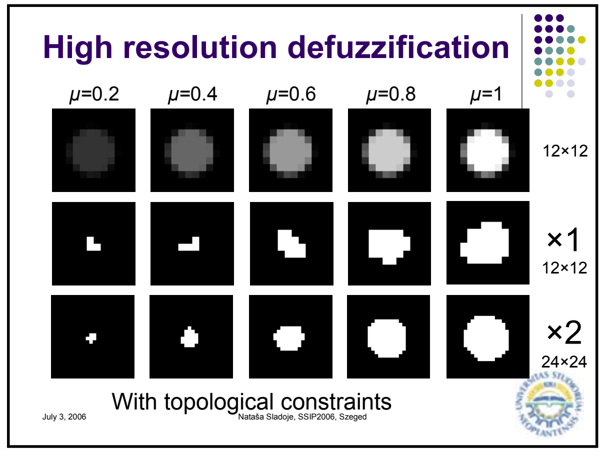


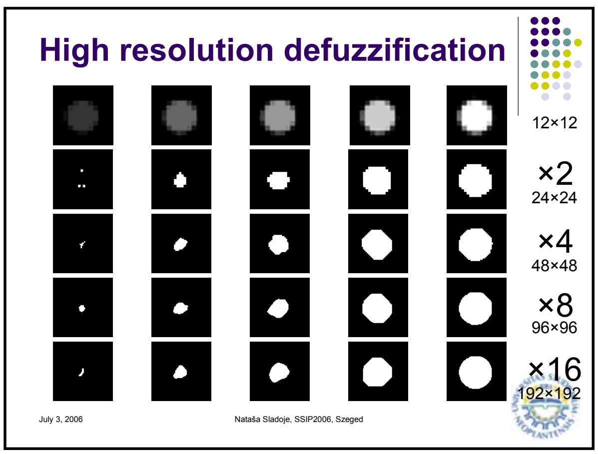
Fuzzy membership is (approximately) 5/16.

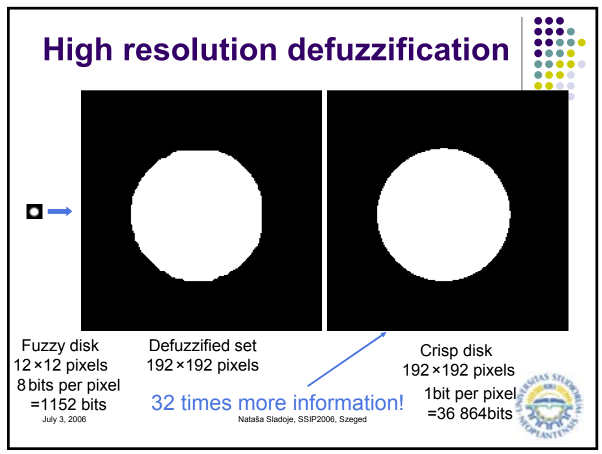
Area of the block of (crisp) subpixels is 5 × 1/16

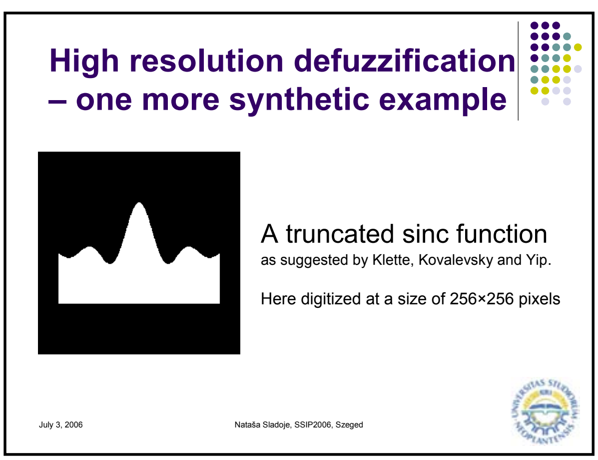
The membership value of a fuzzy pixel corresponds to the area of a block of $r \times r$ crisp sub-pixels.

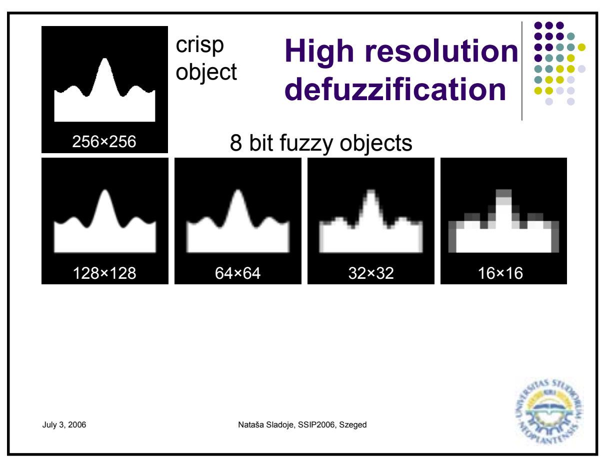

High resolution defuzzification
μ=0.2
μ=0.4
μ=0.6
μ=0.8
μ=1
12×12
×1
12×12
×2
24×24
July 3, 2006
Nataša Sladoje, SSIP2006, Szeged


High resolution defuzzification
μ=0.2
μ=0.4
μ=0.6
μ=0.8
μ=1
12×12
×1
12×12
×2
24×24
With topological constraints
July 3, 2006
Nataša Sladoje, SSIP2006, Szeged

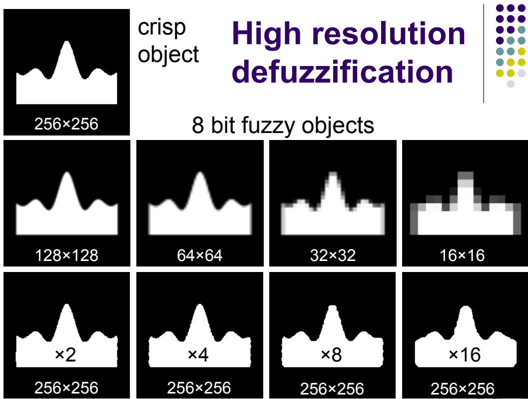

High resolution defuzzification
12×12
×2
24×24
×4
48×48
×8
96×96
×16
192×192
July 3, 2006
Nataša Sladoje, SSIP2006, Szeged


High resolution defuzzification
Fuzzy disk
12×12 pixels
8 bits per pixel
=1152 bits
Defuzzified set
192×192 pixels
Crisp disk
192×192 pixels
1bit per pixel
=36 864bits
32 times more information!
July 3, 2006
Nataša Sladoje, SSIP2006, Szeged

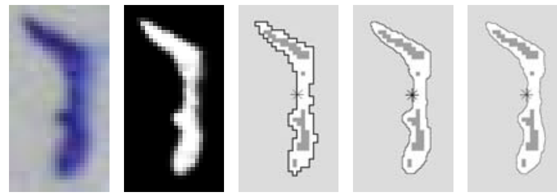

High resolution defuzzification
– one more synthetic example
A truncated sinc function
as suggested by Klette, Kovalevsky and Yip.
Here digitized at a size of 256×256 pixels
July 3, 2006
Nataša Sladoje, SSIP2006, Szeged


crisp
object
High resolution
defuzzification
256×256
8 bit fuzzy objects
128×128
64×64
32×32
16×16
July 3, 2006
Nataša Sladoje, SSIP2006, Szeged

High resolution defuzzification



Realistic examples: Bone



Conclusions: Defizzification

- Defuzzification is a useful crisp segmentation method.
- The suggested defuzzification method consideres both shape geometry and fuzzy membership values of the elements of the original.
- A reconstruction at increased spatial resolution is possible.

Questions to be answered today

- What is a fuzzy set?
- What is a fuzzy segmented image?
- What are the advantages of using them?
- How do we relate fuzzy segmented images to conventional binary images?
- What is defuzzification?
- How do we perform it?
- Any conclusions?

A better solution to a crisp problem is found by looking first in a larger space of fuzzy sets

which has different (usually less) constraints and therefore allows the algorithms more freedom and reduces errors caused by forced crisp answers at intermediate steps.

Conclusion

Fuzzy sets are good for you!!

...and that was all,

thank you!