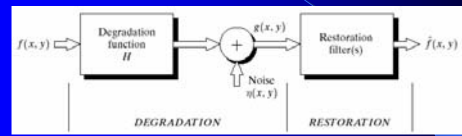
IMAGE RESTORATION

Image restoration

model:

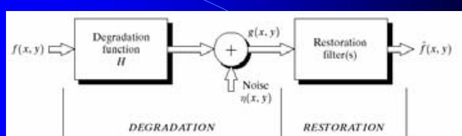


restoration from the degraded image

degradation:

- non-linear mapping: e.g., non-linear sensitivity, image of the straight line is not straight etc.;
- blurring: image of a point is blob;
- moving during the image acquisition;
- probabilistic noise

Restoration

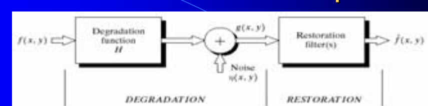


example:

f
original image

g
degraded image

Mathematical description



let us suppose that H is linear (additive, homogeneous), that is,

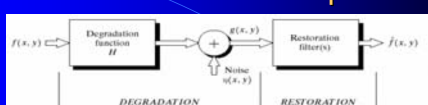
$$H[a_1 \cdot f_1 + a_2 \cdot f_2] = a_1 \cdot H[f_1] + a_2 \cdot H[f_2]$$

then

$$[Hf](x,y) = H \iint f(\xi,\eta) \cdot \delta(x-\xi, y-\eta) d\xi d\eta =$$
$$= \iint H[f(\xi,\eta) \cdot \delta(x-\xi, y-\eta)] d\xi d\eta =$$
$$= \iint f(\xi,\eta) \cdot \underbrace{H[\delta(x-\xi, y-\eta)]}_{h(x,y,\xi,\eta)} d\xi d\eta$$

h is the impulse-spread function or *point-spread function* (PSF) of H

Mathematical description



therefore

$$[Hf](x,y) = \iint f(\xi,\eta) \cdot h(x,y,\xi,\eta) d\xi d\eta$$

integral-equation of the first kind
Fredholm-type

Let us suppose that H is translation invariant:

$$H[f(x-\xi, y-\eta)] = [Hf](x-\xi, y-\eta)$$

then

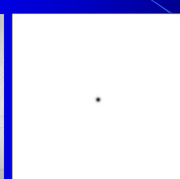
$$H[\delta(x-\xi, y-\eta)] = [H\delta](x-\xi, y-\eta) = h(x-\xi, y-\eta)$$

and

$$Hf = H(f ** \delta) = f ** H\delta = f ** h$$

Example

f
original image



h
point-spread function

$g = f ** h$
degraded image

Restoration

Inverse filtering
least square method
Wiener filter

Jean-Baptiste Joseph Fourier 1768-1830

- taught mathematics in Paris
- eventually traveled to Egypt with Napoleon to become the secretary of the Institute of Egypt
- after fall of Napoleon worked at Bureau of Statistics
- elected to National Academy of Sciences in 1817

- La Theorie Analytique de la Chaleur (The Analytic Theory of Heat), 1822
- revolutionary ideas about how to solve a class of linear differential equations

Fourier transformation (FT)

$f \xrightarrow{\mathcal{F}} F$ $\quad [\mathcal{F}f](X) = F(X) = \int_{-\infty}^{\infty} f(x)\cdot e^{-2\pi i x X}\,dx$

inverse Fourier transformation (IFT)

$f \xleftarrow{\mathcal{F}^{-1}} F$ $\quad [\mathcal{F}^{-1}F](x) = f(x) = \int_{-\infty}^{\infty} F(X)\cdot e^{2\pi i x X}\,dX$

$f \xrightarrow{\mathcal{F}} F$ uniquely determined*

Sums of sinusoids

inverse Fourier transformation (IFT)

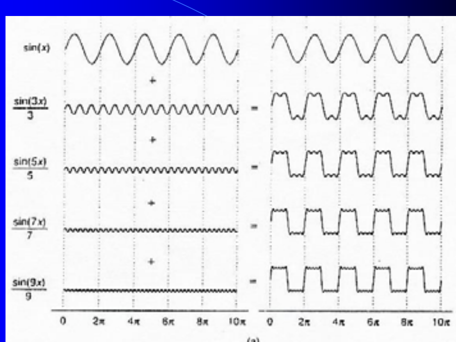
$f \xleftarrow{\mathcal{F}^{-1}} F$ $\quad f(x) = \int_{-\infty}^{\infty} F(X)\cdot e^{2\pi i x X}\,dX$

an interpretation:

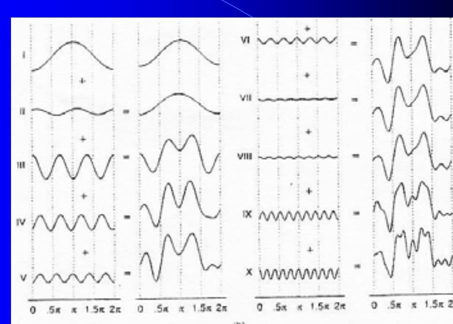
Any periodic function can be decomposed into a series of sinusoidal waveforms of various frequencies and amplitudes.

$$f(x) = \int_{-\infty}^{\infty} F(X)\cdot \cos(2\pi x X)\,dX + i\cdot \int_{-\infty}^{\infty} F(X)\cdot \sin(2\pi x X)\,dX$$

Sums of sinusoids



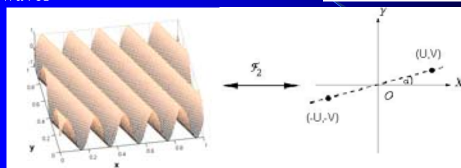
Sums of sinusoids



Relations

$F(X,Y)=\begin{cases}1, & \text{if } (X,Y)=(U,V);\\ 0, & \textit{otherwise}\end{cases}$

waves



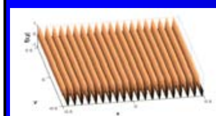
$f(x,y)\approx 1\cdot e^{2\pi i(xU+yV)}+1\cdot e^{-2\pi i(xU+yV)}$
$=2\cdot\cos(2\pi(xU+yV))$

direction $\alpha=\tan^{-1}(U/V)$

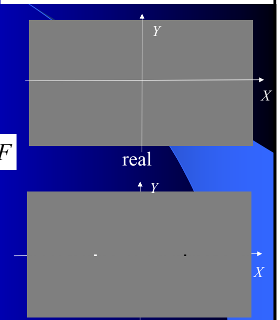
frequency $L=\sqrt{U^2+V^2}$

Sums of sinusoids

$f(x,y)=\sin(40\pi x)$ $\quad f\xrightarrow{F}F\quad$ $F(X,Y)=\frac{i}{2}[\delta(X+20,0)-\delta(X-20,0)]$



$f\xrightarrow{F}F$



Waves, points, and frequencies

points in the image $F(X,Y)$ represent the contribution of frequency (X,Y) to the original image $f(x,y)$

The Fourier transformation determines the magnitude (amplitude – $|F(X,Y)|$) of each possible frequency (X,Y).

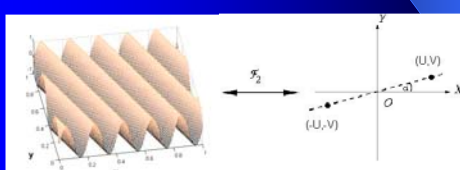
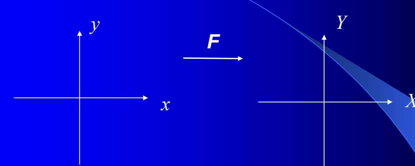


Image and frequency spaces

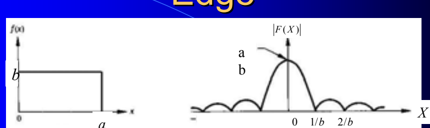


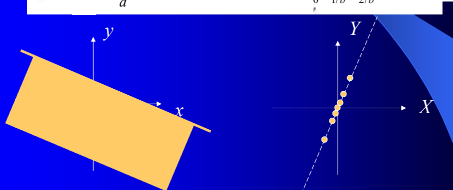
$f\xrightarrow{F}F$

normal space
image space

Fourier space
frequency space

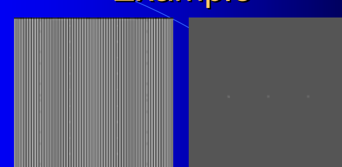
Edge

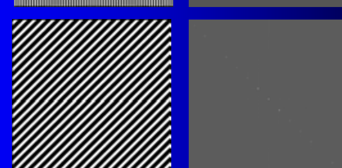


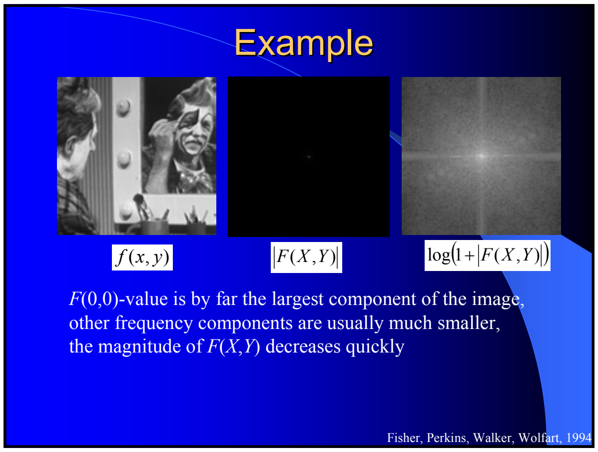


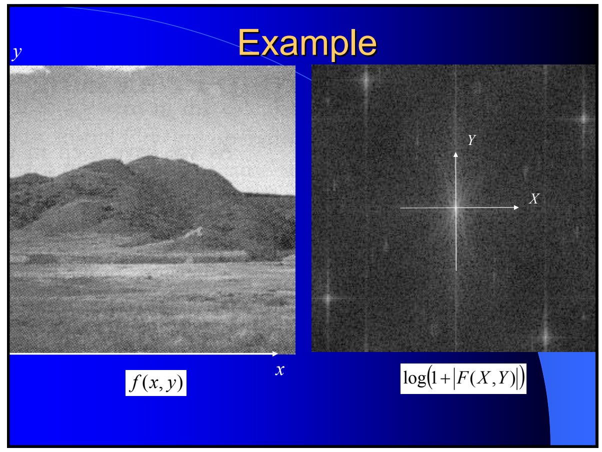
there is a line being perpendicular to the direction of the edge

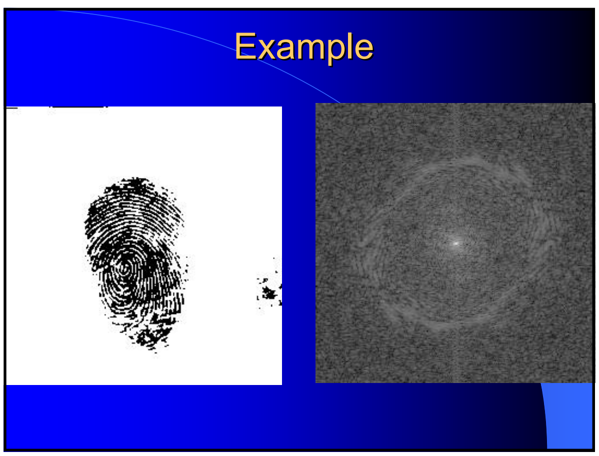
Example

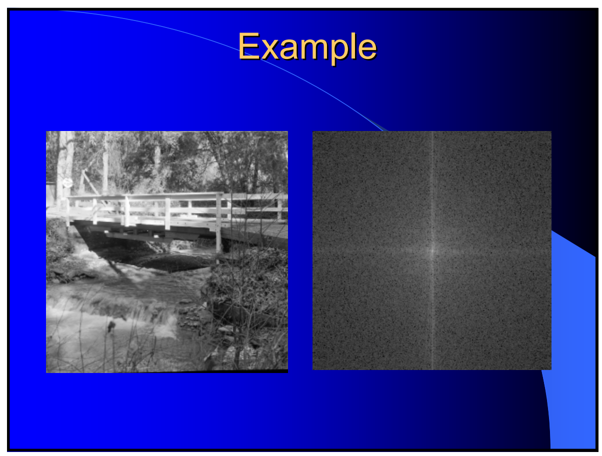


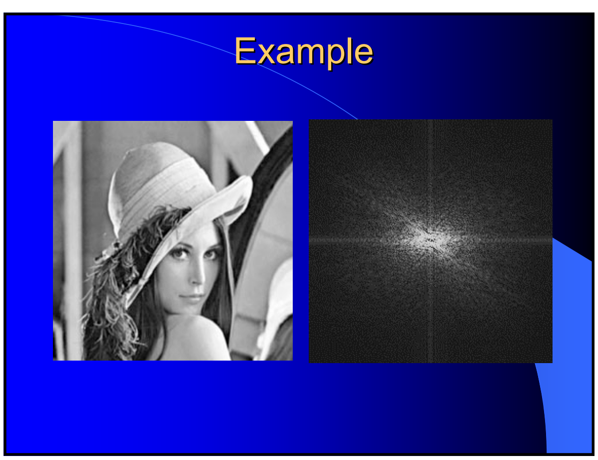


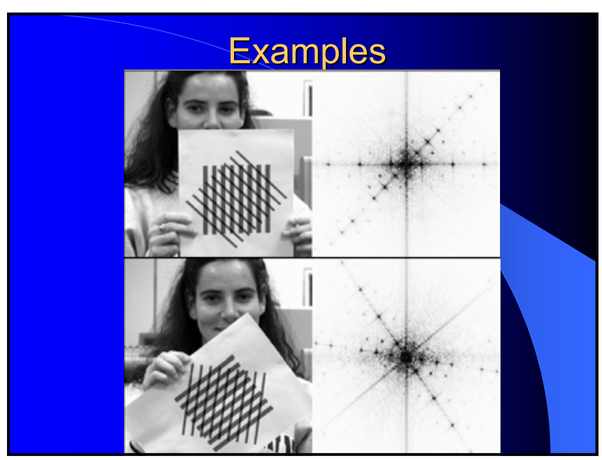

Example
$f(x,y)$
$|F(X,Y)|$
$\log(1+|F(X,Y)|)$
$F(0,0)$-value is by far the largest component of the image,
other frequency components are usually much smaller,
the magnitude of $F(X,Y)$ decreases quickly
Fisher, Perkins, Walker, Wolfart, 1994


Example
y
x
Y
X
$f(x,y)$
$\log(1+|F(X,Y)|)$


Example


Example


Example


Examples

Convolution

$$[f*g](x)=\int_{-\infty}^{\infty} f(\xi)\cdot g(x-\xi)\,d\xi \qquad \text{1D}$$

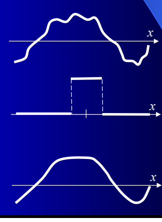
$$[f*g](x,y)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\xi,\eta)\cdot g(x-\xi,y-\eta)\,d\xi\,d\eta \qquad \text{2D}$$

example: smoothing

$f(x)$

$$g(x)=\begin{cases}1, & \text{if } |x|<1/2,\\ 0, & \text{otherwise}\end{cases}$$

$$[f*g](x)=\int_{x-1/2}^{x+1/2} f(\xi)\,d\xi$$



Convolution theorem

$$f*h \xrightarrow{\;\boldsymbol{F}\;} F\cdot H$$

$$f\cdot h \xrightarrow{\;\boldsymbol{F}\;} F*H$$

Filtering

(in the Fourier space) multiplication of F with a filter function H : $F{\cdot}H$

Convolution theorem:

$$f*h \xrightarrow{\;\boldsymbol{F}\;} F\cdot H$$

$$h=\frac{1}{l^2}\begin{pmatrix}1 & \cdots & 1\\ \vdots & & \vdots\\ 1 & \cdots & 1\end{pmatrix}_{l\times l} \xrightarrow{\;\boldsymbol{F}\;}$$

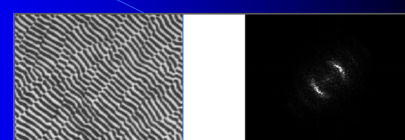
$$H(j,k)=\frac{1}{N}\sum_{m=0}^{l-1}\sum_{n=0}^{l-1}1\cdot e^{-\frac{2\pi i}{N}(mj+nk)}=$$

$$=\frac{1}{N}\left(\sum_{m=0}^{l-1}e^{-\frac{2\pi i}{N}mj}\right)\cdot\left(\sum_{n=0}^{l-1}e^{-\frac{2\pi i}{N}nk}\right)=$$

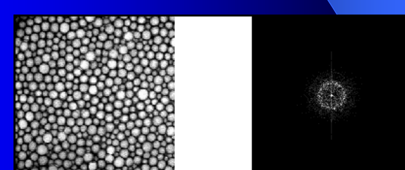
$$=\frac{1}{N}e^{i\pi(l-1)(j+k)}\frac{\sin(\pi lj)}{\sin(\pi j)}\cdot\frac{\sin(\pi kl)}{\sin(\pi k)}$$

Textures

an electron microscope view of the fibers in a metal specimen

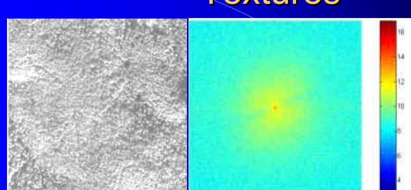


cell structures

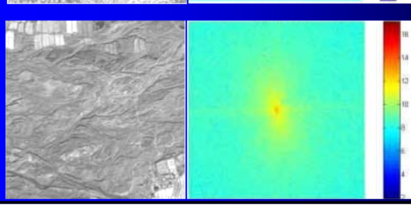


Xin Miao

Textures



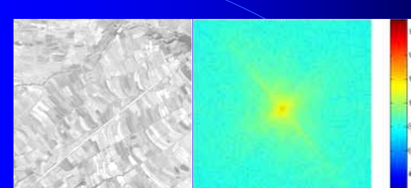
forest



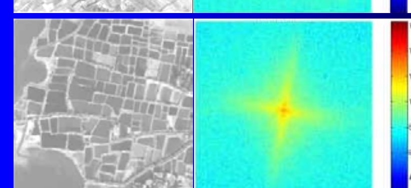
mud

Xin Miao

Textures



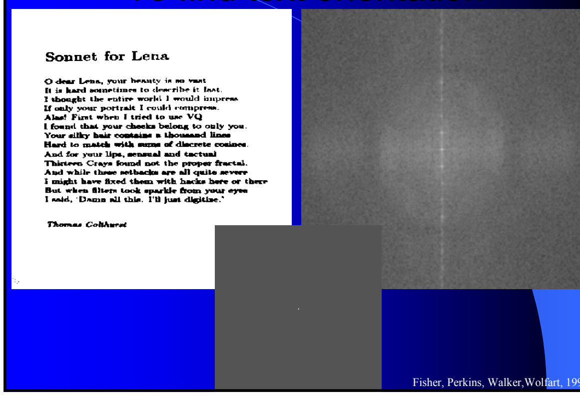
field



pond

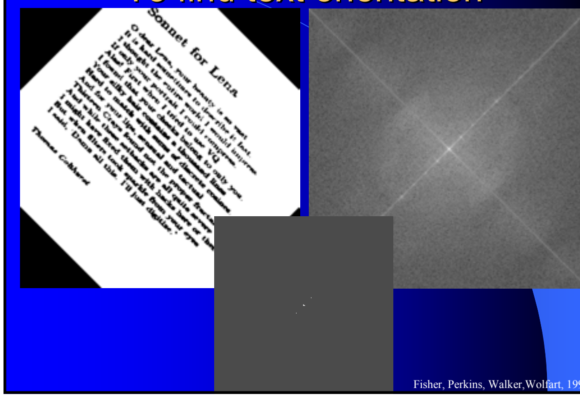
Xin Miao

To find text orientation



Fisher, Perkins, Walker, Wolfart, 1994

To find text orientation



Fisher, Perkins, Walker, Wolfart, 1994

Measurements on the power spectrum

$E(X,Y) = |F(X,Y)|^2$ Fourier power spectrum

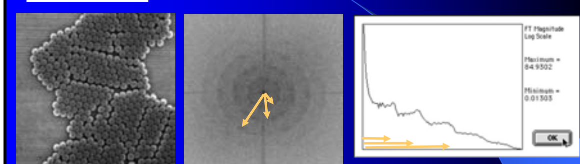
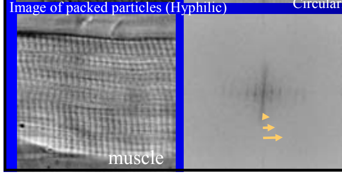


Image of packed particles (Hyphilic)

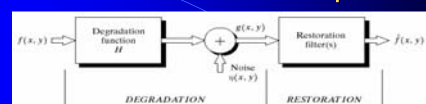
Circularly averaged radial plot of power spectrum



muscle

Reindeer Graphics

Mathematical description



therefore

$$[Hf](x,y) = \iint f(\xi,\eta)\cdot h(x,y,\xi,\eta)\,d\xi\,d\eta$$

integral-equation of the first kind
Fredholm-type

Let us suppose that H is translation invariant:

$$H[f(x-\xi, y-\eta)] = [Hf](x-\xi, y-\eta)$$

then

$$H[\delta(x-\xi, y-\eta)] = [H\delta](x-\xi, y-\eta) = h(x-\xi, y-\eta)$$

and

$$Hf = H(f**\delta) = f**H\delta = f**h$$

Inverse filtering

Inverse filtering: $g = f**h \quad \overset{F}{\longleftrightarrow} \quad G = F\cdot H$

$$f = \mathbf{F}^{-1}\left(\frac{G}{H}\right) \quad \overset{F}{\longleftrightarrow} \quad F = \frac{G}{H}$$

Difficulties:

- What is in the points (X,Y), where $H(X,Y) \approx 0$?
  Let us filter these points and their neighborhood (band filters).
- What about noisy images?
  $g = f**h + n \qquad G = F\cdot H + N.$
  Inverse filtering:
  $$\hat{F} = \frac{G}{H} = \frac{F\cdot H + N}{H} = F + \frac{N}{H}$$
  if H is small then N/H is large

Inverse filtering

$$\hat{F} = \frac{G}{H} = \frac{F\cdot H + N}{H} = F + \frac{N}{H}$$

experience:
H: quickly decreasing function,
N: not (so quickly) decreasing
let us cut the high frequencies

remark: if the noise is known (generally it is not) then

$$F = \frac{G}{H} - \frac{N}{H}$$

Example

translation with constant speed in T time

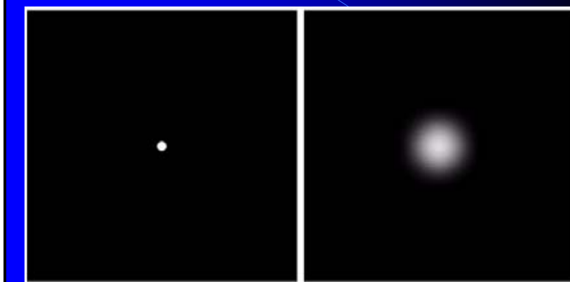
f
original image

h
point-spread function

$g = f ** h$
degraded image

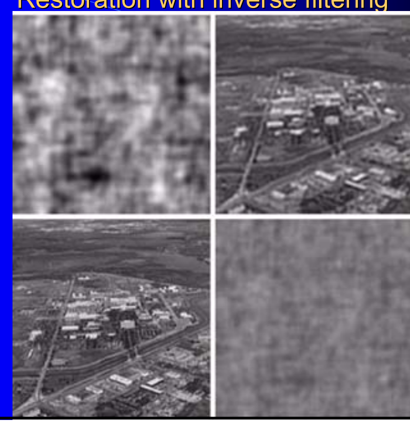
$$F = \frac{G}{H} - \frac{N}{H} \quad \Longrightarrow \quad \frac{G}{H} = \frac{F \cdot H + N}{H} = F + \frac{N}{H}$$

Model of a point-source

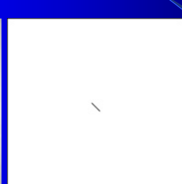


Blurring

Restoration with inverse filtering



Inverse filtering



Least square methods

What is f such that

$$\varepsilon(f) = \|g - Hf\|^2 = \sum_{i=0}^{M-1} (g_i - (Hf)_i)^2 =$$

$$= (g - Hf)^T \cdot (g - Hf) \qquad \text{minimal ?}$$

$$\frac{\partial \varepsilon}{\partial f} = \underbrace{0 = -2H^T(g - Hf)}_{\Downarrow}$$

$$\hat{f} = \underbrace{(H^T H)^{-1} H^T}_{R} g$$

$$\hat{f} = H^{-1} g, \qquad \text{if } H \text{ square matrix and invertable}$$

Least square methods

Let Q be a linear oprator (matrix),
look for an f such that

$$\varepsilon(f) = \|Qf\|^2 = (Qf)^T \cdot (Qf) \qquad \text{minimal}$$

let us suppose that $\|g - Hf\|^2 = \|n\|^2$ (constrain)

$$J(f) = (Qf)^T (Qf) + \alpha \cdot \left((g - Hf)^T (g - Hf) - n^T n \right)$$

$$\underbrace{\frac{\partial J}{\partial f} = 0 = 2Q^T Qf - 2\alpha H^T (g - Hf)}_{\Downarrow}$$

$$\hat{f} = \underbrace{\left(H^T H + \frac{1}{\alpha} Q^T Q \right)^{-1} \cdot H^T}_{R} g$$

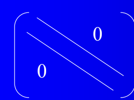
Covariance matrix

$$S_f = M\left((F - M(F)) \cdot (F - M(F))^T \right) = $$
$$= M\left(F \cdot F^T \right), \quad \text{if } M(F) = 0;$$
$$S_n = M\left((N - M(N)) \cdot (N - M(N))^T \right) = $$
$$= M\left(N \cdot N^T \right), \quad \text{if } M(N) = 0;$$

ij-component:
correlation between
the elements *i* and *j*

S_f: image energy spectrum, S_n: noise energy spectrum
symmetric matrices, 1-s in the diagonal

typical:



there is positive correlation between the nearly elements,
even more, it can be supposed that the correlation depends only on the distances between the image points

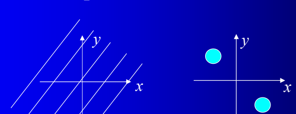
Covariance matrix

additive noise: $g = f + n$ $\qquad G(X,Y) = F(X,Y) + N(X,Y)$

e.g. $n(x,y) = A \cdot \sin(x_0 x + y_0 y)$,

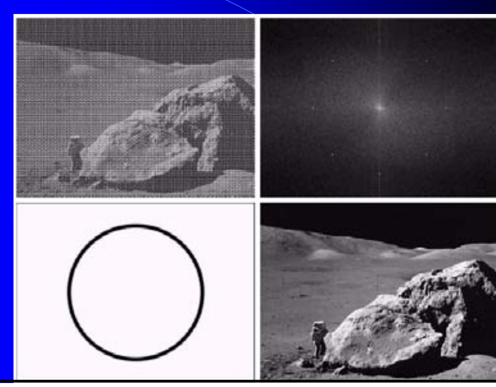
then

$$N(X,Y) = -\frac{iA}{2} \cdot \left[\delta\left(X - \frac{x_0}{2\pi}, Y - \frac{y_0}{2\pi} \right) - \delta\left(X + \frac{x_0}{2\pi}, Y + \frac{y_0}{2\pi} \right) \right]$$

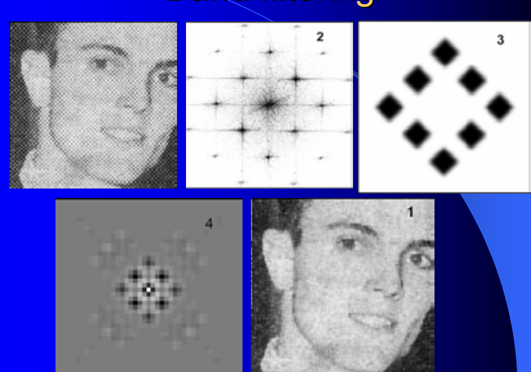


restoration of $f(x,y)$ by subtraction if n or N is known,
if not then let us try to find the places of the impulses in image G and let us use a proper band-pass filter for restoration

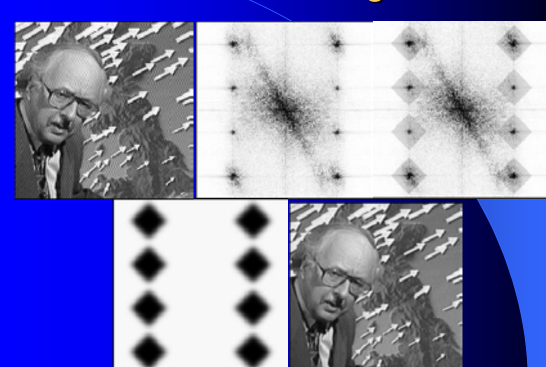
Band filtering

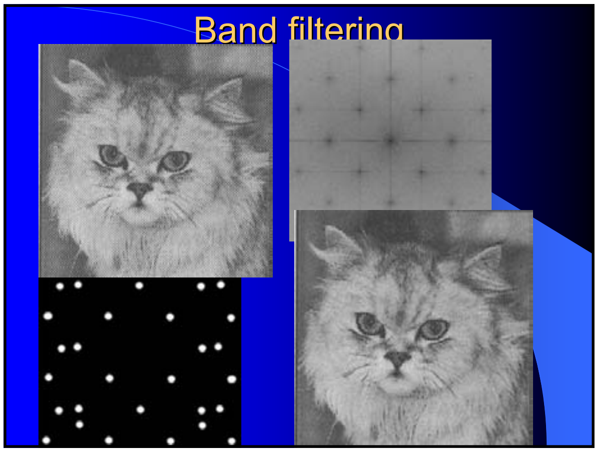


Band filtering



Band filtering

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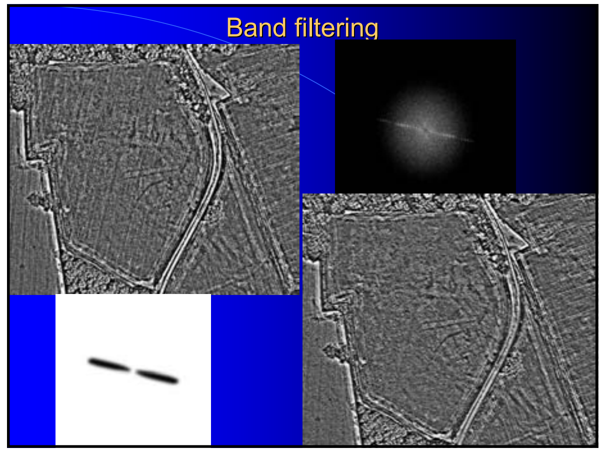
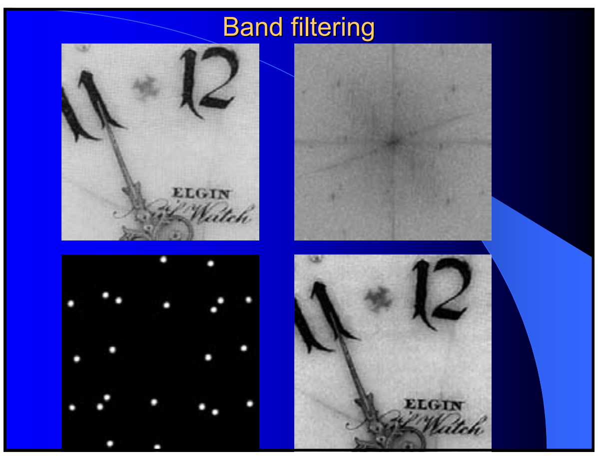
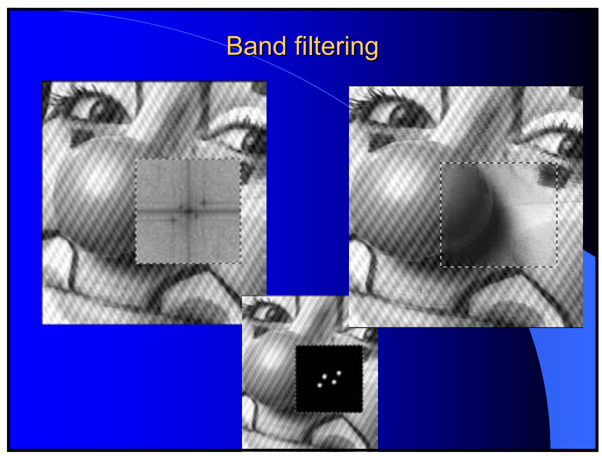
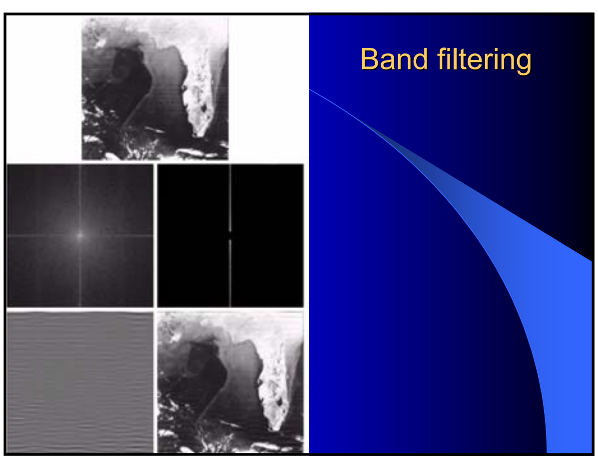
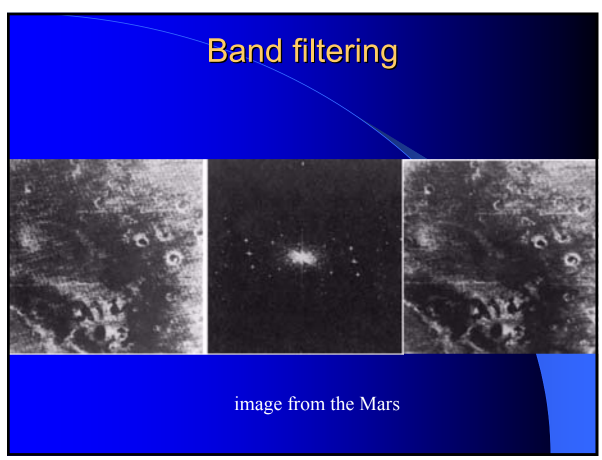

Band filtering


Band filtering


Band filtering


Band filtering


Band filtering
image from the Mars

Wiener filtering

$$\hat{f} = \left(H^T H + \frac{1}{\alpha} Q^T Q\right)^{-1} \cdot H^T g$$

if $Q = I$, then

$$\hat{f} = \left(H^T H + \frac{1}{\alpha}\right)^{-1} \cdot H^T g$$ $\alpha = 0$: pseudo-inverse filter

ha $Q^T Q = S_f^{-1} S_n$, akkor

$$\hat{f} = \left(H^T H + \frac{1}{\alpha} S_f^{-1} S_n\right)^{-1} \cdot H^T g$$ Wiener filter with parameter

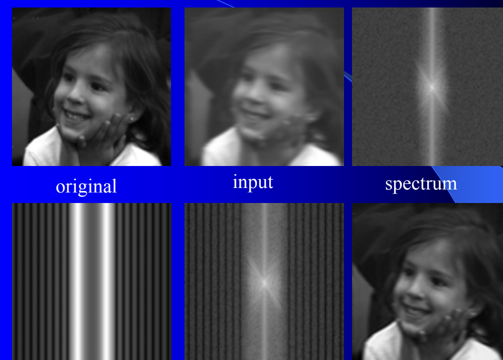
properties:

if there is no noise ($S_n = O$), then it is an inverse filter;

if $\alpha = 1$, then it is optimal w.r.t. $M(\|f - \hat{f}\|)$

(supposing that $M(s^T \cdot n) = O$, i.e., there is no correlation between the image and the noise)

Wiener filter



original / input / spectrum

Wiener filter / spectrum of the result / result

Wiener filter

original / input / result

Inverse filter and Wiener filter

inverse filter without freq. cut. / inverse filter with freq.cut. / Wiener filter