

16th Summer School on Image Processing, 9 July, 2008, Vienna, Austria

Discrete Tomography



Péter Balázs

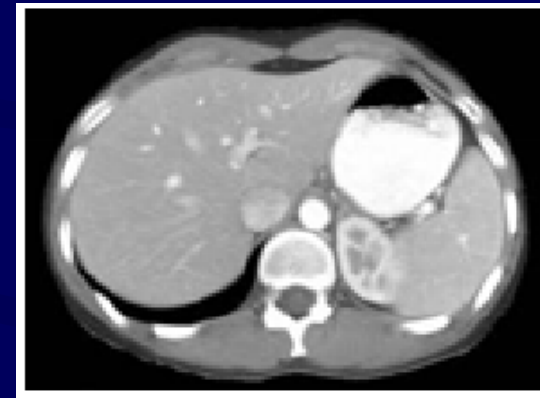
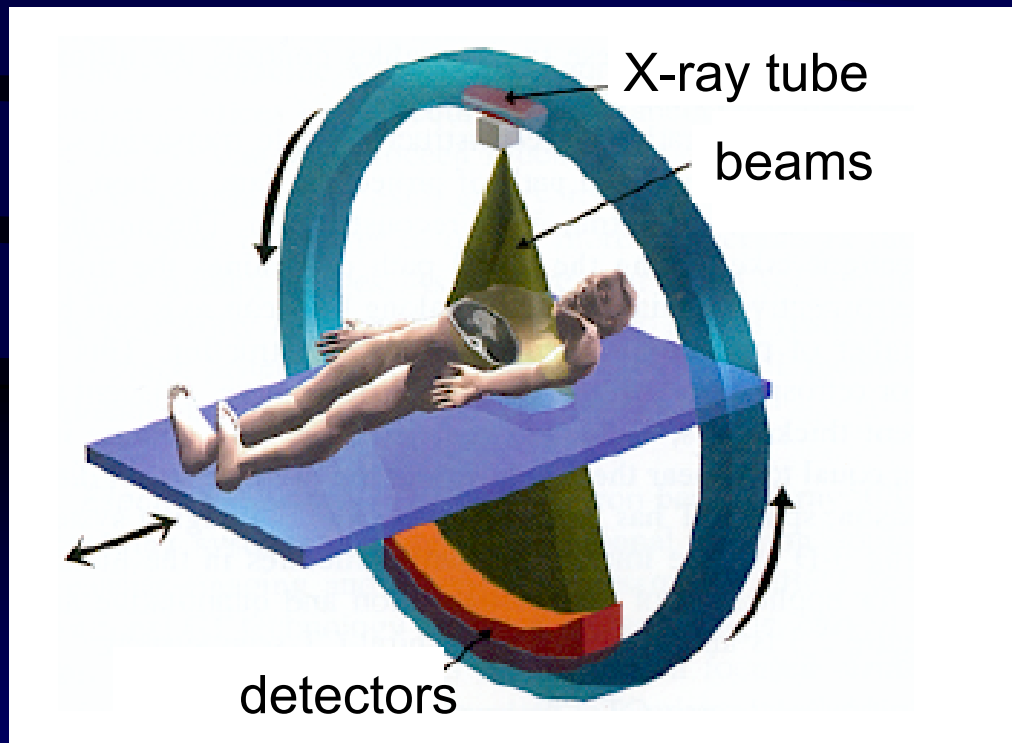
Department of Image Processing and
Computer Graphics
University of Szeged, HUNGARY

Outline

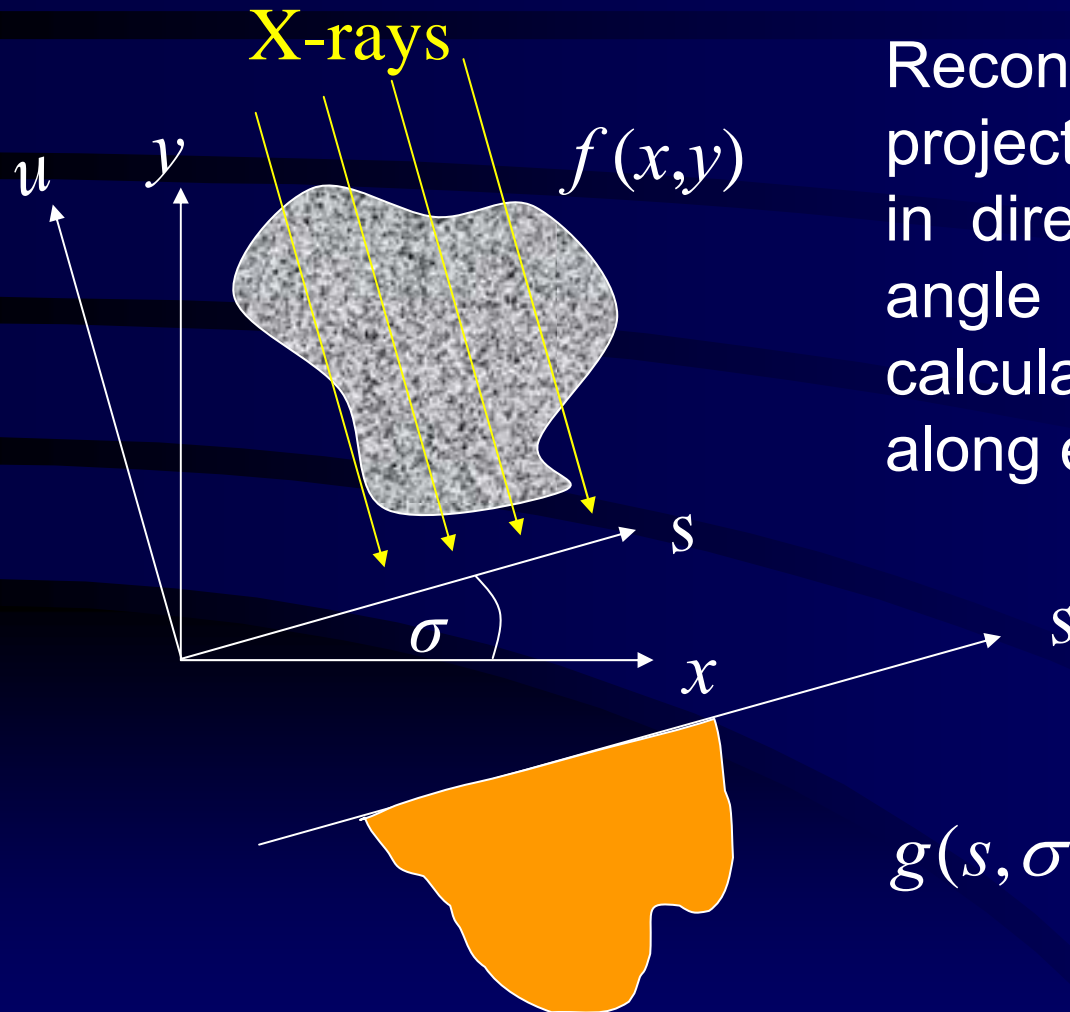
- Computerized Tomography
- Discrete and Binary Tomography
- Binary Tomography using 2 projections
- Ambiguity and complexity problems
- A priori information
- Reconstruction as optimization
- Applications

Computerized Tomography

- A technique for imaging the 2D cross-sections of 3D objects (usually human parts)



The Mathematics of CT

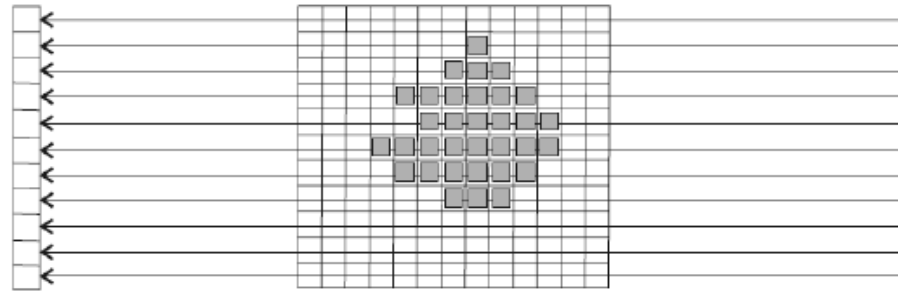


Reconstruct $f(x,y)$ from its projections where a projection in direction u (defined by the angle σ) can be obtained by calculating the line integrals along each line parallel to u .

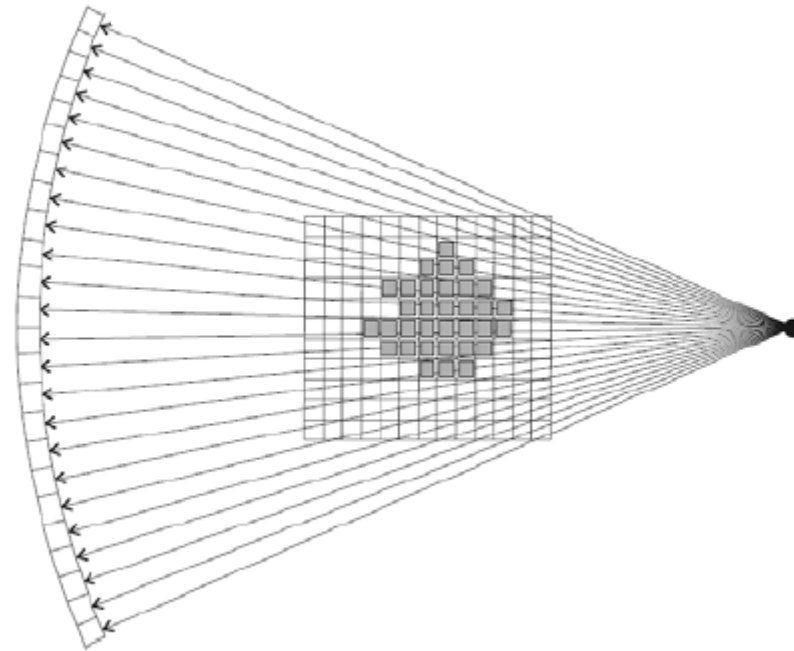
$$g(s, \sigma) = \int_{-\infty}^{\infty} f(x, y) du$$

Projection geometries

Parallel

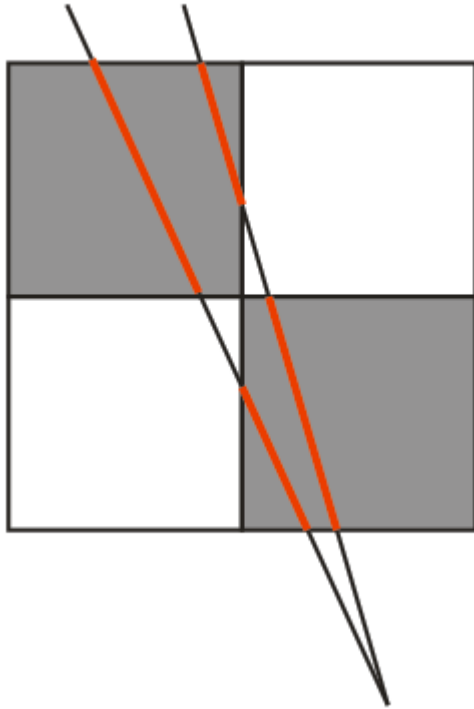


Fan beam

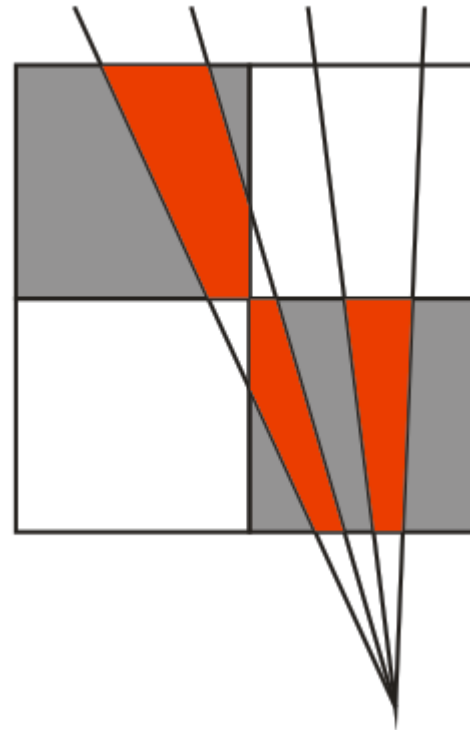


Projections

Line integrals



Area integrals



Discrete Tomography

- In CT we need a few hundred projections
 - time consuming
 - expensive
 - may damage the object
- In certain applications the range of the function to be reconstructed is discrete and known → DT (only few (2-10) projections are needed)

KNOWING THE DISCRETE RANGE



# projs.	Conv. method	Discretized image	DT method
8			
12			
16			

L. Ruskó, A.K., Z. Kiss, L. Rodek, 2003

9



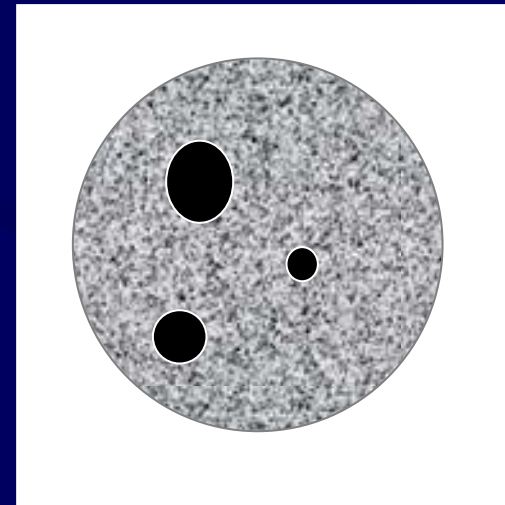
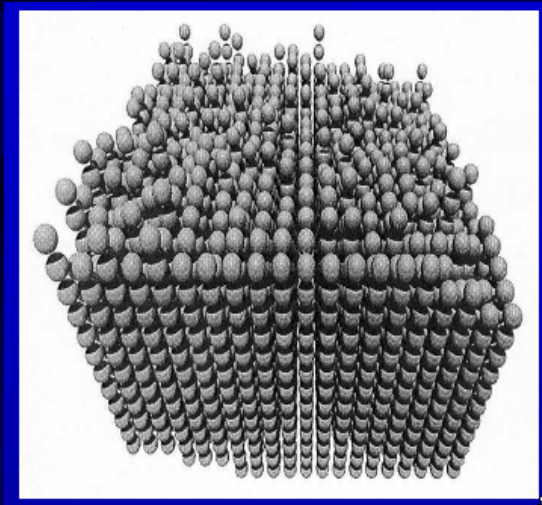
Source: Attila Kuba

8

Binary Tomography

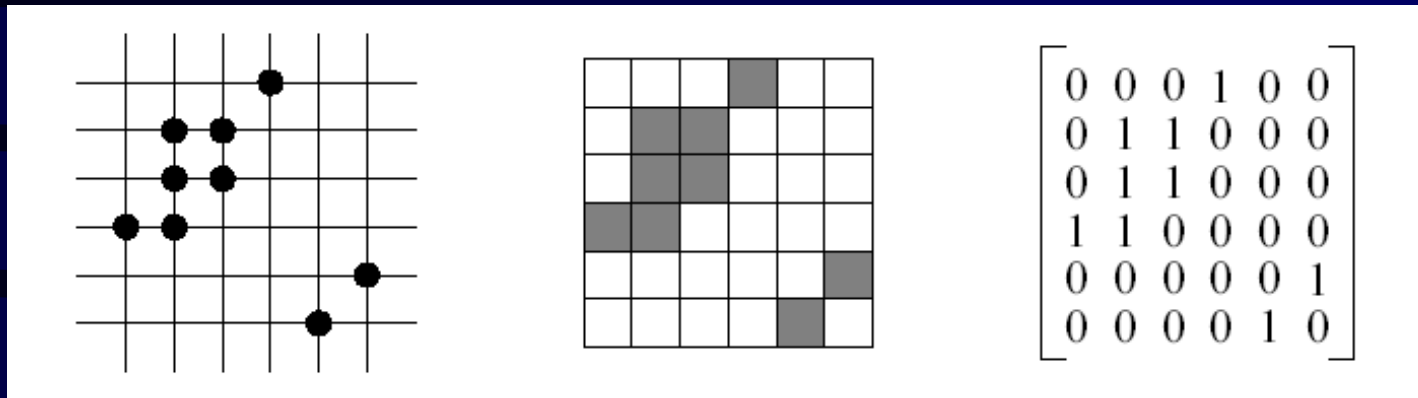
the range of the function to be reconstructed is $\{0,1\}$
(absence or presence of material)

- **angiography**: parts of human body with X-rays
- **electron microscopy**: structure of molecules or crystals
- **non-destructive testing**: obtaining shape information of homogeneous objects

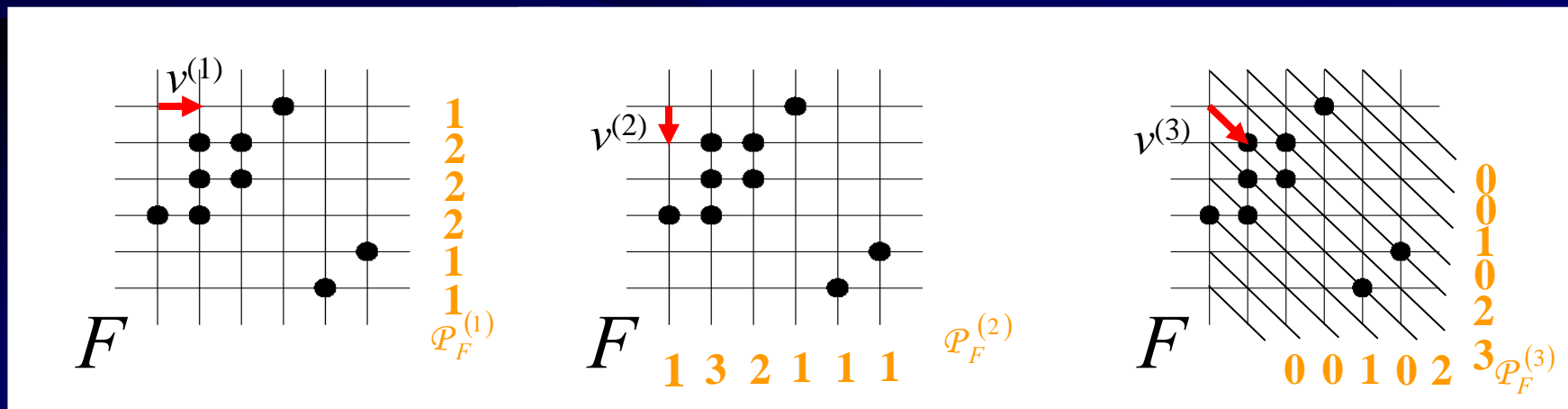


Discrete Sets and Projections

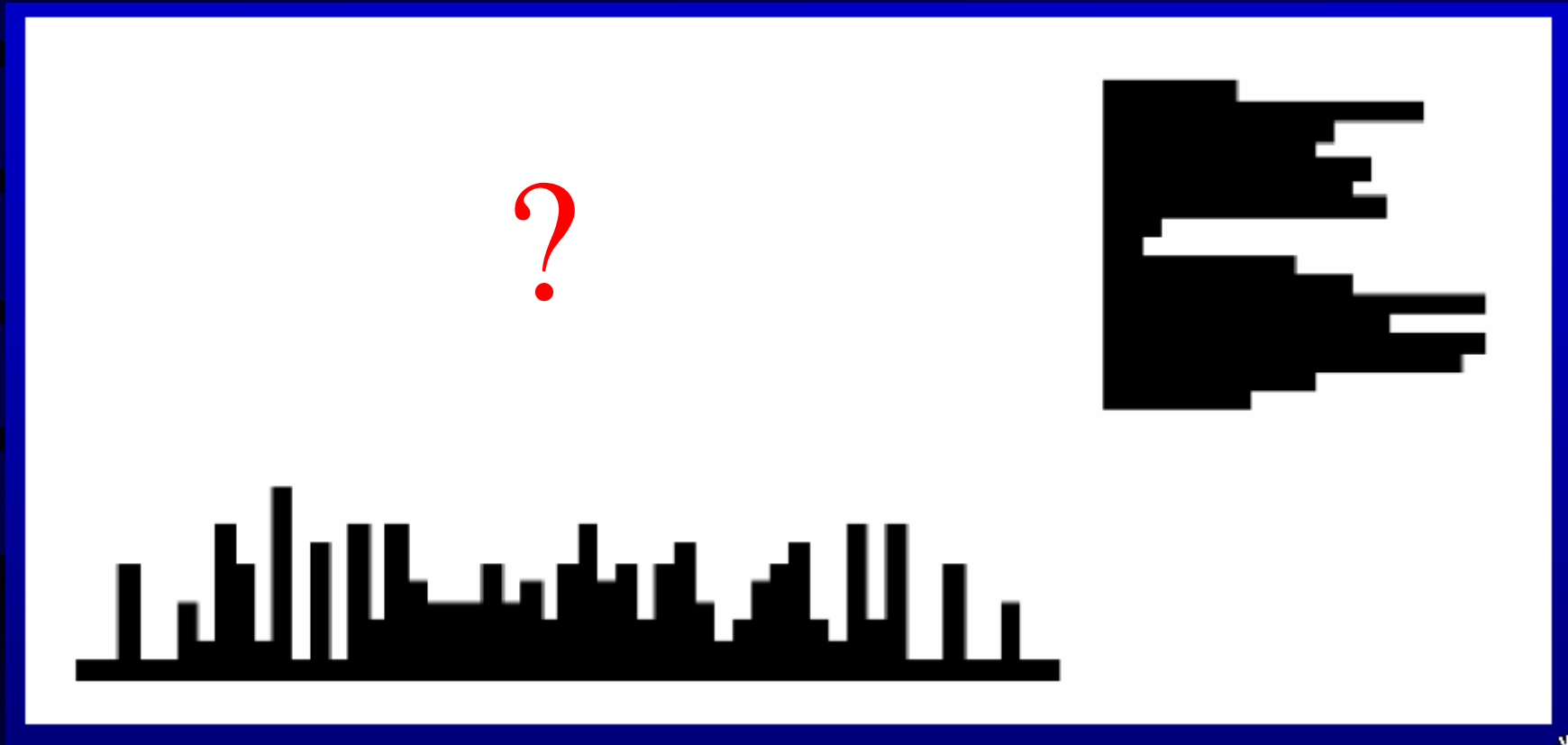
- **discrete set:** a finite subset of \mathbb{Z}^2



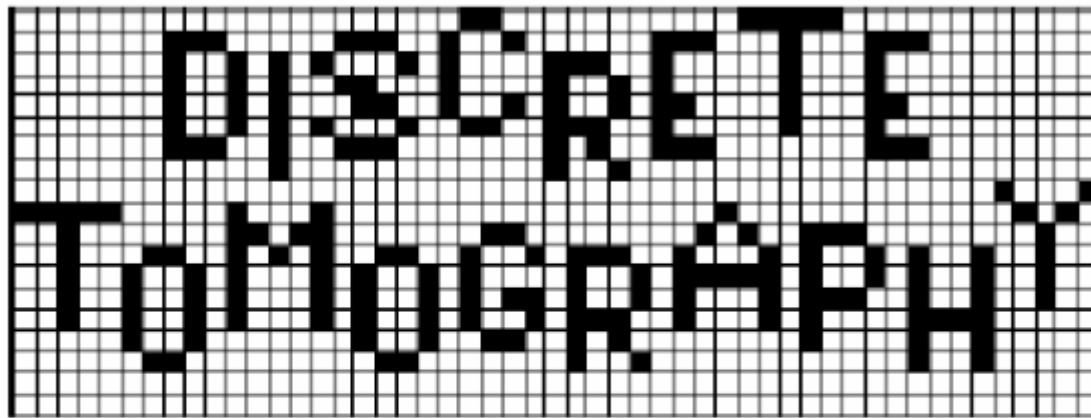
- reconstruct a discrete set from its projections



Reconstruction from 2 Projections



Reconstruction from 2 Projections



11

Example for Uniqueness

3			
2			
1			
	3	2	1

3	1	1	1
2			
1			
	3	2	1

3	1	1	1
2	1		
1	1		
	3	2	1

3	1	1	1
2	1	1	
1	1		
	3	2	1

unique

Example for Inconsistency

3			
3			
1			
	3	3	1

3	1	1	1
3			
1			
	3	3	1

3	1	1	1
3	1	1	1
1			
	3	3	1

inconsistent

Classification

3			
3			
1			
	3	3	1

inconsistent

3	1	1	1
2	1	1	
1	1		
	3	2	1

unique

1	1	
1		1
	1	1

1		1
1	1	
	1	1

non-unique

Main Problems

Consistency: Does there exist a discrete set with a given set of projections.

Uniqueness: Is a discrete set uniquely determined by a given set of projections.

Reconstruction: Construct a discrete set from its projections.

Reconstruction \rightarrow Consistency

Uniqueness and Switching Components

configuration

1	
	1

	1
1	

2		1	1			
4		1	1	1	1	
3	1	1		1		
4	1	1	1			1
1	1					
	3	4	3	2	1	1

2		1	1			
4		1	1	1		1
3	1	1		1		
4	1	1	1		1	
1	1					
	3	4	3	2	1	1

The presence of a switching component is necessary and sufficient for non-uniqueness

Reconstruction

Ryser, 1957 – from row sums R and column sums S

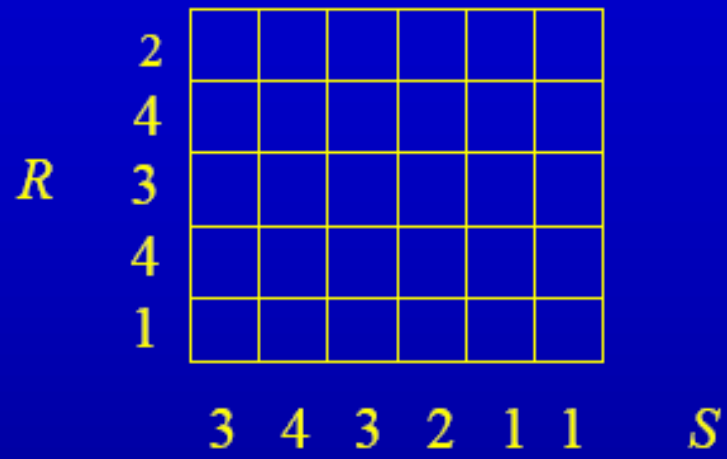
Order the elements of S in a non-increasing way by $\pi \rightarrow S'$

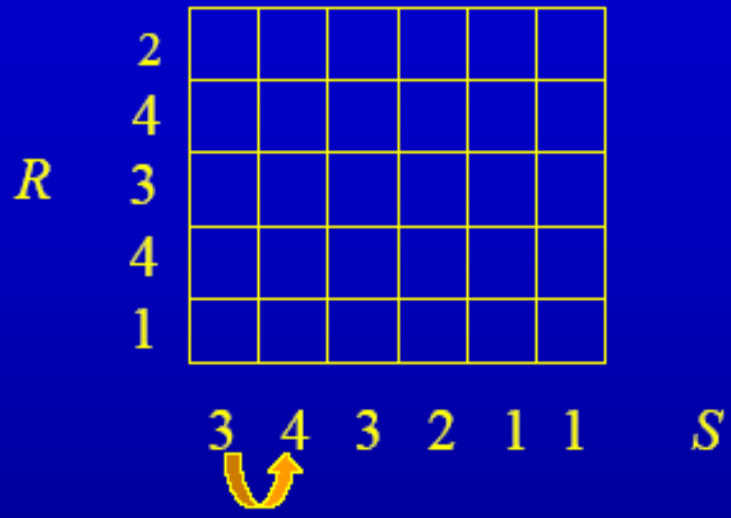
Fill the rows from left to right $\rightarrow B$ (canonical matrix)

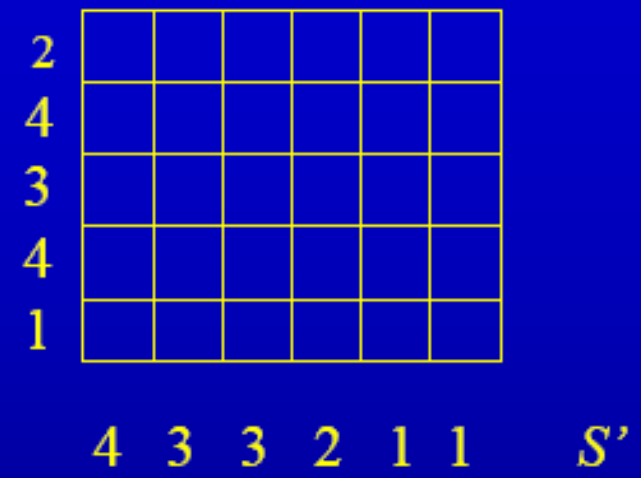
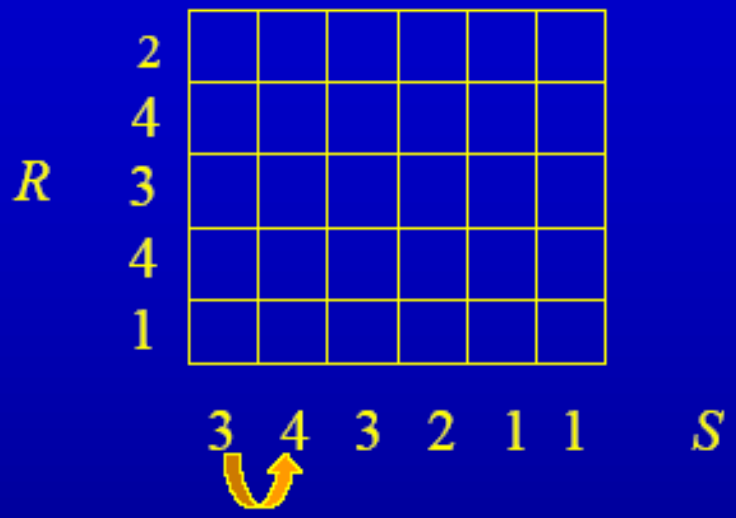
Shift elements from the rightmost columns of B to the columns where $S(B) < S'$

Reorder the columns by applying the inverse of π

Complexity: $O(nm + n \log n)$








R

2					
4					
3					
4					
1					

3 4 3 2 1 1 *S*



2					
4					
3					
4					
1					

4 3 3 2 1 1 *S'*

R

2	1	1			
4	1	1	1	1	
3	1	1	1		
4	1	1	1	1	
1	1				

=*B*

5 4 3 2 0 0 *S(B)*

4 3 3 2 1 1 *S'*

R

2					
4					
3					
4					
1					

3 4 3 2 1 1 *S*

2					
4					
3					
4					
1					

4 3 3 2 1 1 *S'*

R

2	1	1			
4	1	1	1	1	
3	1	1	1		
4	1	1	1	1	
1	1				

=*B*

5 4 3 2 0 0 *S(B)*

4 3 3 2 1 1 *S'*

R

2					
4					
3					
4					
1					

3 4 3 2 1 1 *S*

2					
4					
3					
4					
1					

4 3 3 2 1 1 *S'*

R

2	1	1			
4	1	1	1	1	
3	1	1	1		
4	1	1	1	1	
1	1				

=*B*

5 4 3 2 0 0 *S(B)*
 4 3 3 2 1 1 *S'*

2	1	1			
4	1	1	1		1
3	1	1	1		
4	1	1	1	1	
1	1				

5 4 3 1 0 1 *S(B)*
 4 3 3 2 1 1 *S'*

R

2					
4					
3					
4					
1					

3 4 3 2 1 1 *S*

2					
4					
3					
4					
1					

4 3 3 2 1 1 *S'*

R

2	1	1			
4	1	1	1	1	→
3	1	1	1		
4	1	1	1	1	
1	1				

=*B*

5 4 3 2 0 0 *S(B)*

4 3 3 2 1 1 *S'*

2	1	1			
4	1	1	1		1
3	1	1	1		
4	1	1	1	1	→
1	1				

5 4 3 1 0 1 *S(B)*

4 3 3 2 1 1 *S'*

R

2	1	1				
4	1	1	1			1
3	1	1	1			
4	1	1	1		1	
1	1					

5 4 3 0 1 1 $S(B)$

4 3 3 2 1 1 S'

R

2	1	1				
4	1	1	1	→		1
3	1	1	1	→		
4	1	1	1		1	
1	1					

5	4	3	0	1	1	$S(B)$
4	3	3	2	1	1	S'

R

2	1	1				
4	1	1	1	→		1
3	1	1	1	→		
4	1	1	1		1	
1	1					

5 4 3 0 1 1

S(B)

4 3 3 2 1 1

S'

2	1	1				
4	1	1		1		1
3	1	1		1		
4	1	1	1		1	
1	1					

5 4 1 2 1 1

S(B)

4 3 3 2 1 1

S'

R

2	1	1				
4	1	1	1	→		1
3	1	1	1	→		
4	1	1	1			1
1	1					

5 4 3 0 1 1 *S(B)*
 4 3 3 2 1 1 *S'*

2	1	1				
4	1	1		1		1
3	1	1		1		
4	1	1	1			1
1	1					

5 4 1 2 1 1 *S(B)*
 4 3 3 2 1 1 *S'*

R

2	1	1	→			
4	1	1	→	1		1
3	1	1		1		
4	1	1	1			1
1	1					

5 4 1 2 1 1 *S(B)*
 4 3 3 2 1 1 *S'*

R

2	1	1				
4	1	1	1	→		1
3	1	1	1	→		
4	1	1	1			1
1	1					

5 4 3 0 1 1

S(B)

4 3 3 2 1 1

S'

2	1	1				
4	1	1		1		1
3	1	1		1		
4	1	1	1			1
1	1					

5 4 1 2 1 1

S(B)

4 3 3 2 1 1

S'

R

2	1	1	→			
4	1	1	→	1		1
3	1	1		1		
4	1	1	1			1
1	1					

5 4 1 2 1 1

S(B)

4 3 3 2 1 1

S'

2	1		1			
4	1		1	1		1
3	1	1		1		
4	1	1	1			1
1	1					

5 2 3 2 1 1

S(B)

4 3 3 2 1 1

S'

R

2	1	1			
4	1	1	1		1
3	1	1	1		
4	1	1	1		1
1	1				

5 4 3 0 1 1
4 3 3 2 1 1

S(B)
S'

2	1	1			
4	1	1		1	1
3	1	1		1	
4	1	1	1		1
1	1				

5 4 1 2 1 1
4 3 3 2 1 1

S(B)
S'

R

2	1	1			
4	1	1	1		1
3	1	1		1	
4	1	1	1		1
1	1				

5 4 1 2 1 1
4 3 3 2 1 1


S(B)
S'

2	1		1		
4	1		1	1	1
3	1	1		1	
4	1	1	1		1
1	1				

5 2 3 2 1 1
4 3 3 2 1 1


S(B)
S'

R	2		1	1				
	4	1		1	1		1	
	3	1	1		1			
	4	1	1	1			1	
	1	1						
		4	3	3	2	1	1	$S(B)$
		4	3	3	2	1	1	S'



2		1	1			
4	1		1	1		1
<i>R</i> 3	1	1		1		
4	1	1	1		1	
1	1					

4	3	3	2	1	1	$S(B)$
4	3	3	2	1	1	S'



R

2		1	1			
4	1		1	1		1
3	1	1		1		
4	1	1	1		1	
1	1					

4 3 3 2 1 1 $S(B)$

4 3 3 2 1 1 S'

2	1		1			
4		1	1	1		1
3	1	1		1		
4	1	1	1		1	
1		1				

3 4 3 2 1 1 S

Consistency

- Necessary condition: compatibility

$$\sum_{i=1}^m r_i = \sum_{j=1}^n s_j$$

$$r_i \leq n \ (i = 1, \dots, m), \ s_j \leq m \ (j = 1, \dots, n)$$

- Gale, Ryser, 1957: there exist a solution iff

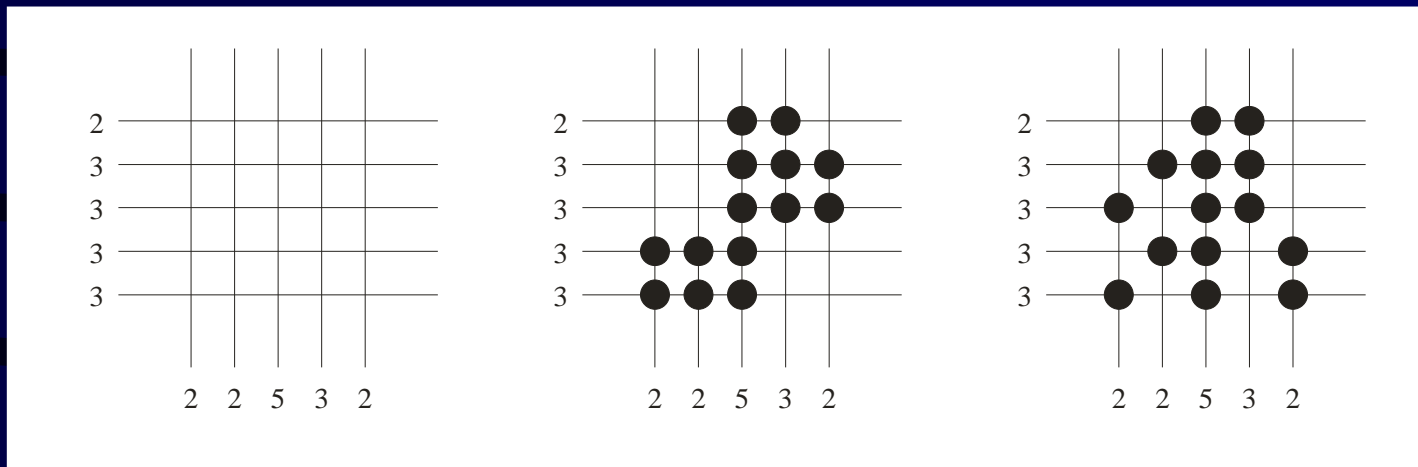
$$\sum_{j=1}^k s'_j \leq \sum_{j=1}^k s(B)_j \quad k = 1, \dots, n$$

3				
3				
1				
	3	3	1	S'
	6			

3	1	1	1	
3	1	1	1	
1	1			
	3	2	2	$S(B)$
	5			

Ambiguity

Due to the presence of switching components there can be many solutions with the same two projections



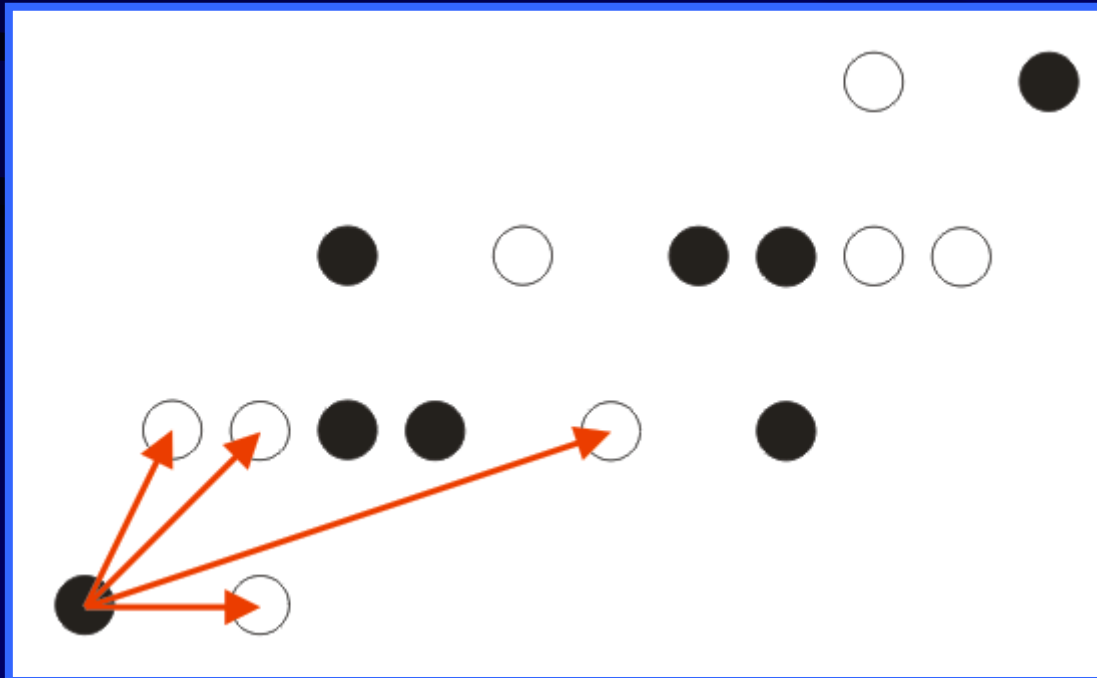
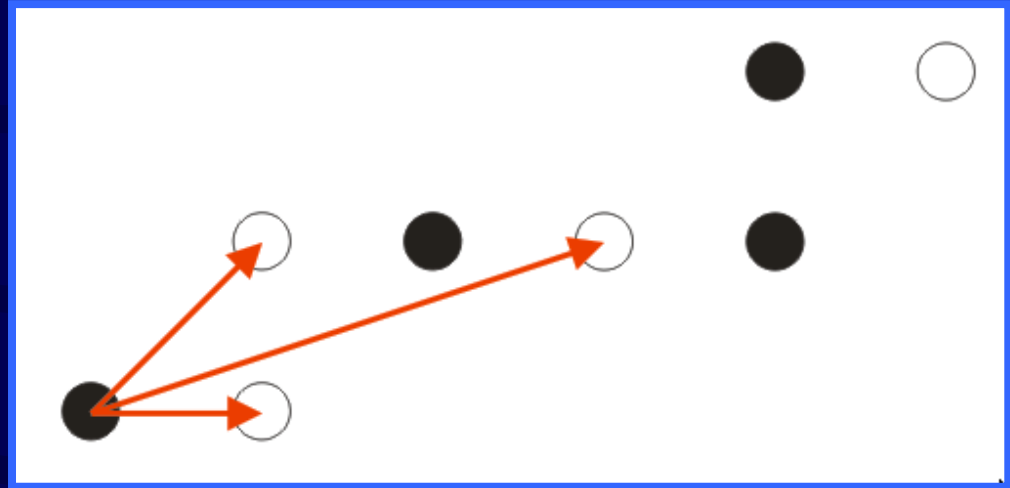
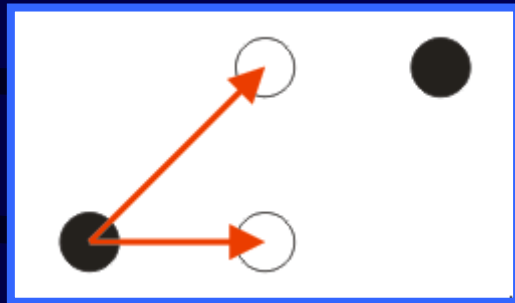
Suggestions:

1. Take further projections along different lattice directions
2. Use a priori information of the set to be reconstructed

Suggestion 1

- In the case of more than 2 projections uniqueness, consistency and reconstruction problems are in general NP-hard – Gardner, Gritzmann 1999
- For an arbitrary number of projections there might be different discrete sets having the same projections

Proof



...

Convexity

	1	1			
	1	1	1	1	1
1	1				
1	1	1			
1					

h-convex

	1				
	1		1	1	1
1	1	1			
1	1	1			
1					

v-convex

	1				
	1	1	1	1	1
1	1	1			
1	1	1			
1					

hv-convex

h-convex or *v-convex*: NP-complete - Barcucci et al., 1996

hv-convex: NP-complete - Woeginger, 1996

Connectedness

	1	1			
	1	1	1		1
1	1			1	
1	1	1			
1					

not 4-connected
but 8-connected

	1				
	1		1	1	1
1	1	1	1		
1	1	1			
1					

4-connected

4-connected: NP-complete - Woeginger, 1996

h-convex or *v*-convex, 4-connected: NP-complete - Barcucci *et al.*, 1996

hv-Convex and Connected Sets

hv-convex 8-connected:

	1				
	1	1	1	1	1
1					
1					
1					

- Chrobak, Dürr, 1999
 $O(mn \cdot \min\{m^2, n^2\})$

hv-convex 4-connected:

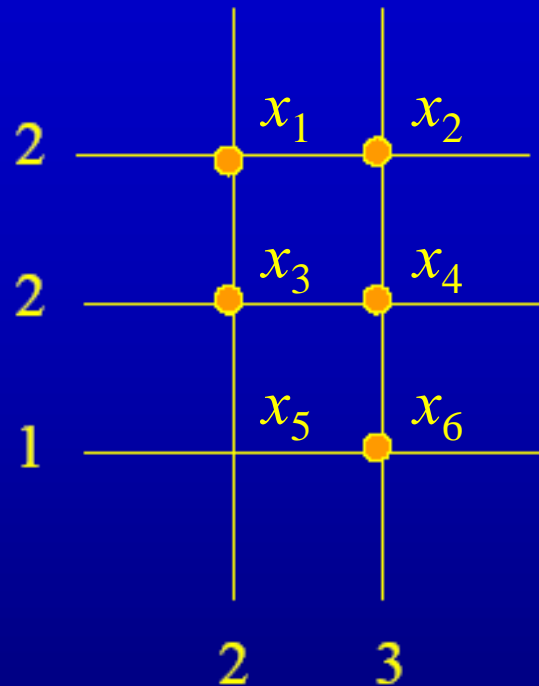
	1				
	1	1	1	1	1
1	1	1			
1	1	1			
1					

- Kuba, 1999
 $O(mn \cdot \min\{m^2, n^2\})$

hv-convex 8- but not 4-connected:

- Balázs, Balogh, Kuba, 2005
 $O(mn \cdot \min\{m, n\})$

Reconstruction as Optimization



$$\left. \begin{array}{rcl}
 x_1 + x_2 & = & 2 \\
 x_3 + x_4 & = & 2 \\
 x_5 + x_6 & = & 1 \\
 x_1 + x_3 + x_5 & = & 2 \\
 x_2 + x_4 + x_6 & = & 3
 \end{array} \right\} b$$

Px

$$P = \begin{pmatrix} 1 & 1 & & & & \\ & & 1 & 1 & & \\ & & & & 1 & 1 \\ 1 & & 1 & & 1 & \\ & 1 & & 1 & & 1 \end{pmatrix} \quad x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix}$$

Optimization

$$Px = b \quad x \in \{0,1\}^{m \times n}$$

Problems:

- binary variables
- big system
- underdetermined (#equations \ll #unknowns)
- inconsistent (if there is noise)

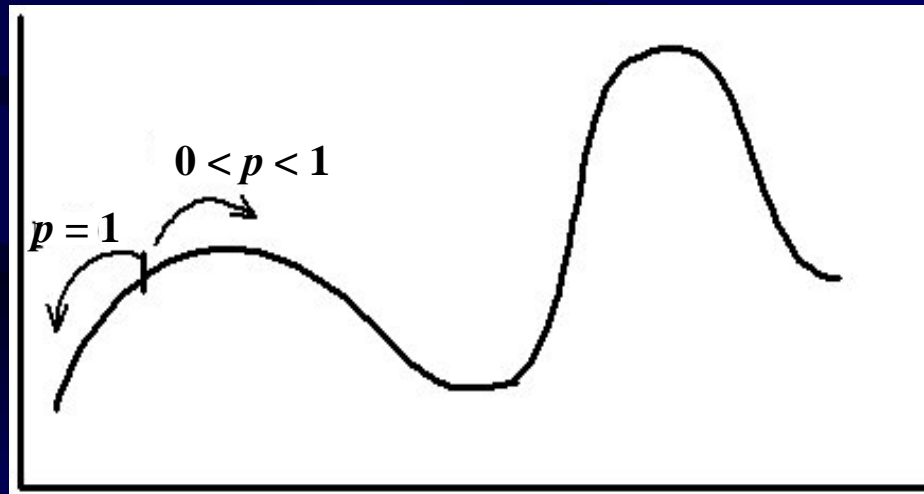
$$x \in \{0,1\}^{m \times n}$$

$$C(x) = \|Px - b\|^2 + g(x) \rightarrow \min$$

Term for prior information: convexity, similarity to a model image, etc.

Solving the Optimization Task

- **Problem:** Classical hill-climbing algorithms can become trapped in local minima.
- **Idea:** Allow some changes that increase the objective function.



Simulated Annealing

- **Annealing**: a thermodynamical process in which a metal cools and freezes.
- Due to the thermal noise the energy of the liquid in some cases grows during the annealing .
- By carefully controlling the cooling temperature the fluid freezes into a minimum energy crystalline.
- **Simulated annealing**: a random-search technique based on the above observation.

Outline of SA

Set initial solution x and temperature T_0

Modify $x_{\text{act}} \rightarrow x'$

Calculate $C(x')$

$C(x') < C(x_{\text{act}})$?

Y

$x_{\text{act}} = x'$

N

$x_{\text{act}} = x'$ with probability $p = e^{-\Delta C/T}$

Termination?

N

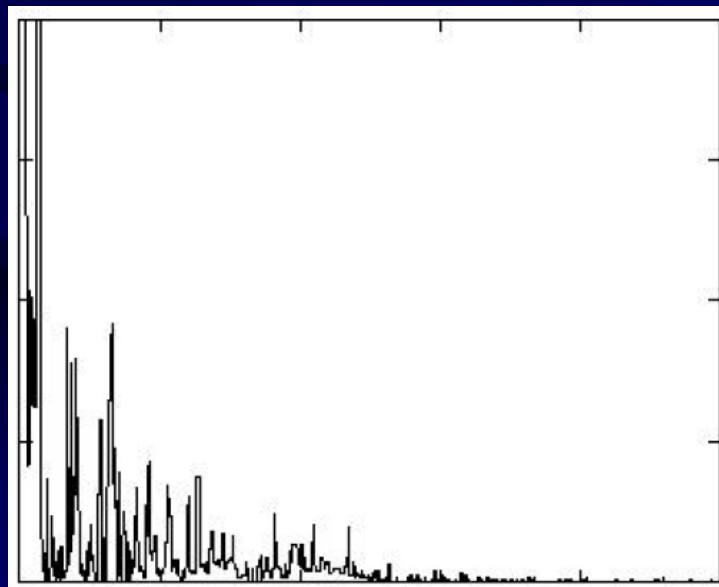
Modify $x_{\text{act}} \rightarrow x'$

Lower temperature

Y
Stop

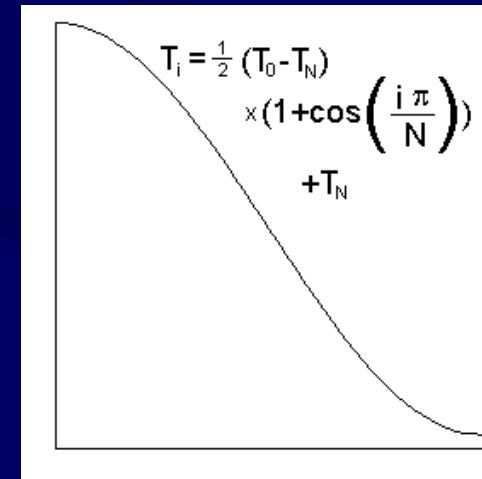
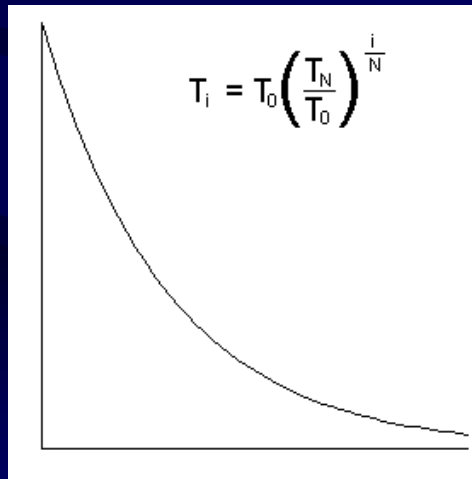
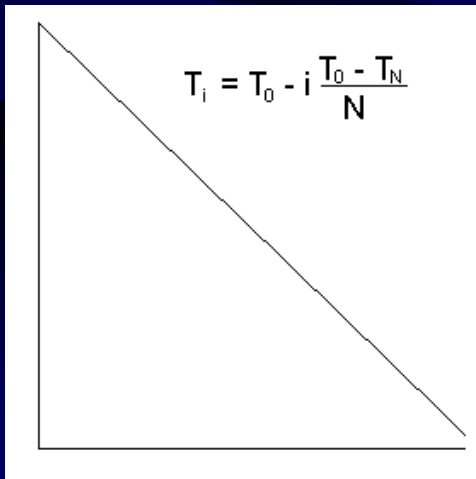
Finding the optimum

- Tuning the parameters appropriately SA finds the global optimum
- Fine-tuning of the parameters for a given optimization problem can be rather delicate



Parameterization of SA

- Initial temperature: T_0
- Stopping criteria: e.g. T_N
- Cooling schedule



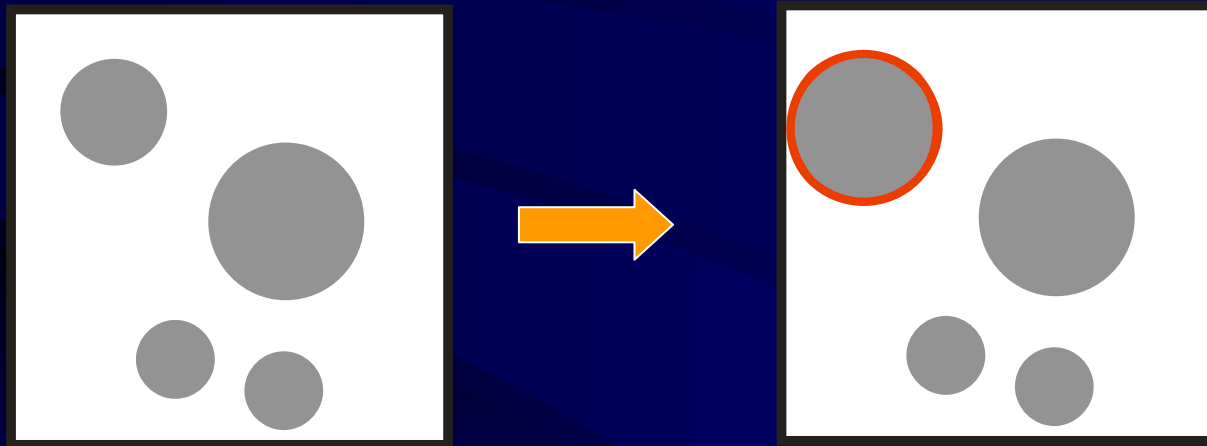
SA in Pixel Based Reconstruction

- A binary matrix describes the binary image
- Randomly invert matrix value(s)

$$\begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}$$

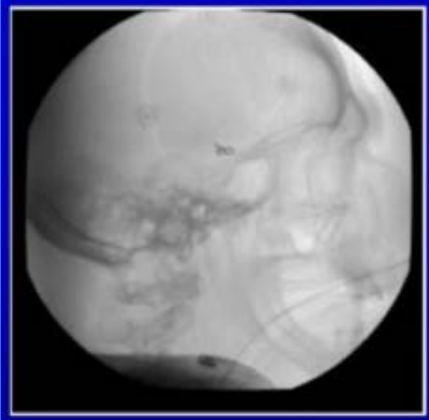
SA in Geometry Based Reconstruction

- The binary image is described by parameters of geometrical objects, e.g. (x,y,r)
- Randomly modify parameter(s) of object(s)

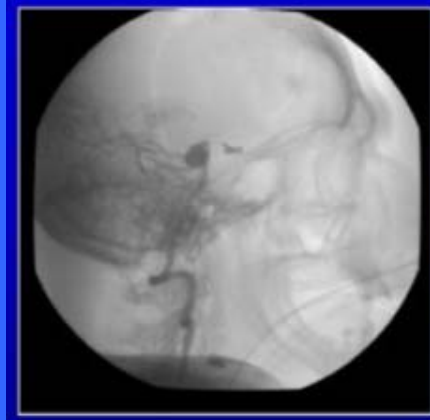


$[(16,53,17), (44,35,25), (26,13,12), (43,8,12)] \rightarrow [(13,50,23), (44,35,25), (26,13,12), (43,8,12)]$

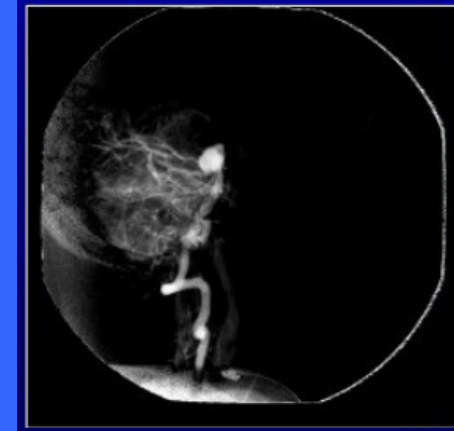
Angiography



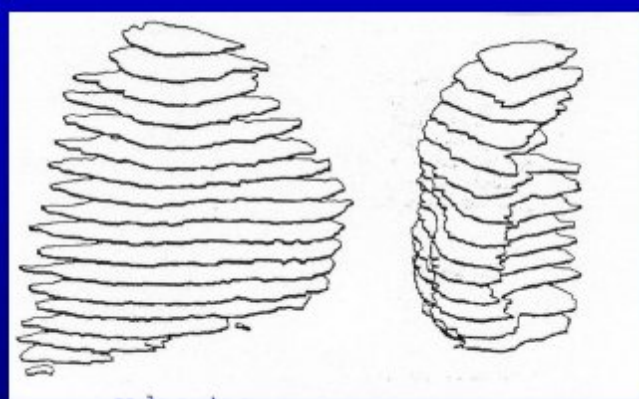
—



=



Heart chambers



Blood vessels



Sources: D.G.W. Onnasch, G.P.M.
Prause, T. Schüle

Neighbouring Slices

Slices which are close to each other in space or time are similar

previous slice

	1	1	1		
	1	1	1	1	
		1	1		

cost matrix

8	7	6	7	8	9
7	4	3	4	5	8
7	4	2	2	4	7
9	8	4	4	5	8
9	9	7	7	8	9

$$x \in \{0,1\}^{m \times n}$$

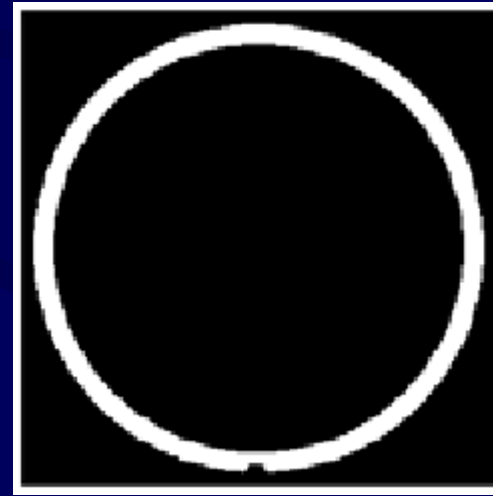
$$C(x) = \|Px - b\|^2 + \sum_{i,j} c_{ij} x_{ij} \rightarrow \min$$

Non-destructive testing

- Pipe corrosion and deposit study
 - 32 fan beam projections



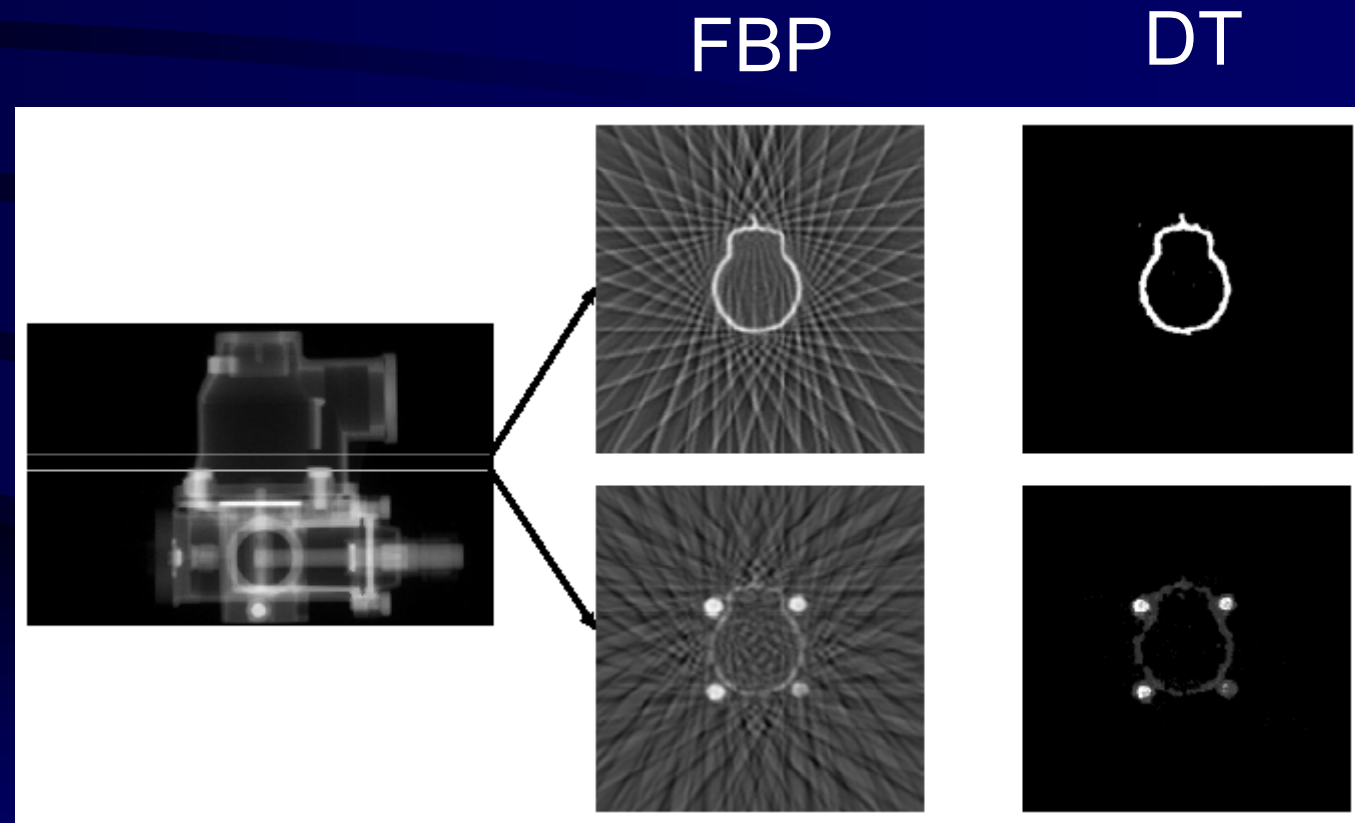
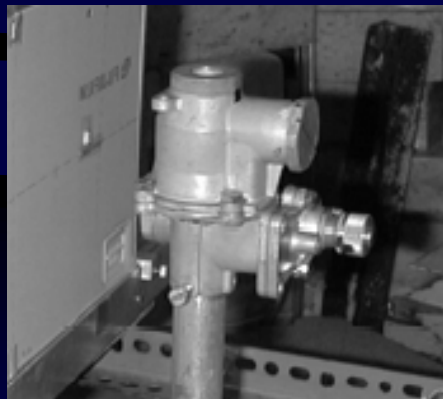
no noise



10 % Gaussian noise

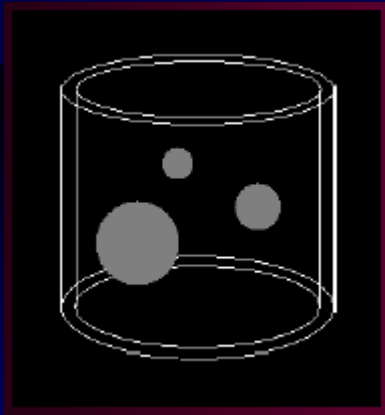
Neutron Tomography I.

- Gas pressure controller
 - 18 projections, pixel based

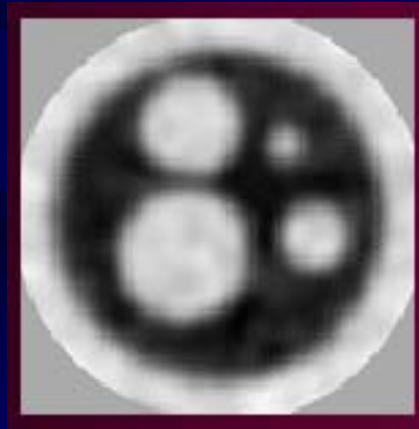


Neutron Tomography II.

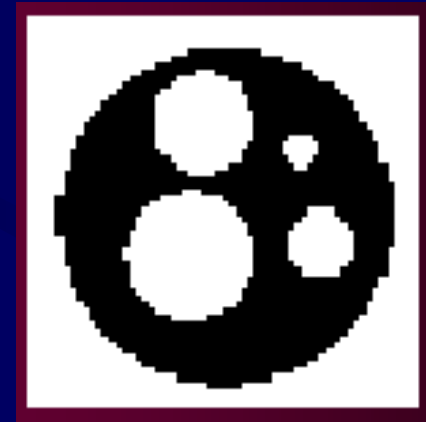
- Reconstruction of disks (air bubbles)
 - 4 projections, geometry based



FBP 60 proj.

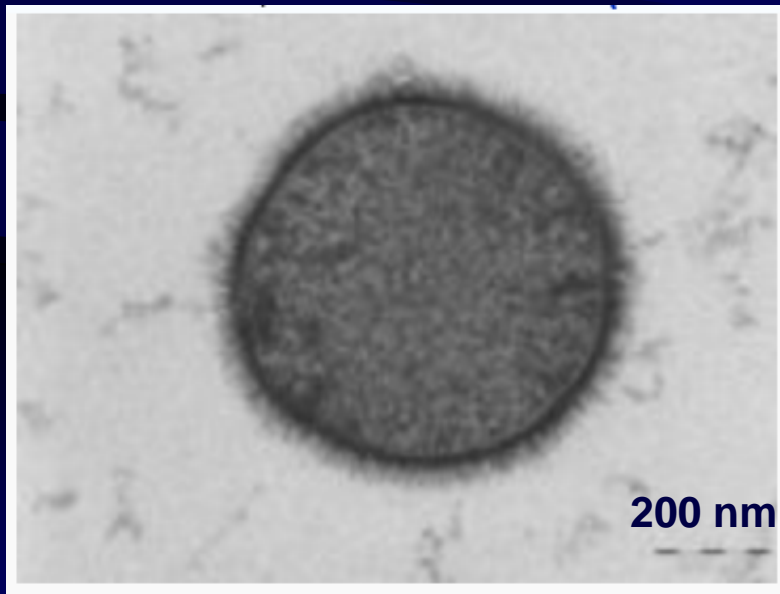


DT 4 proj.



Electron Microscopy I.

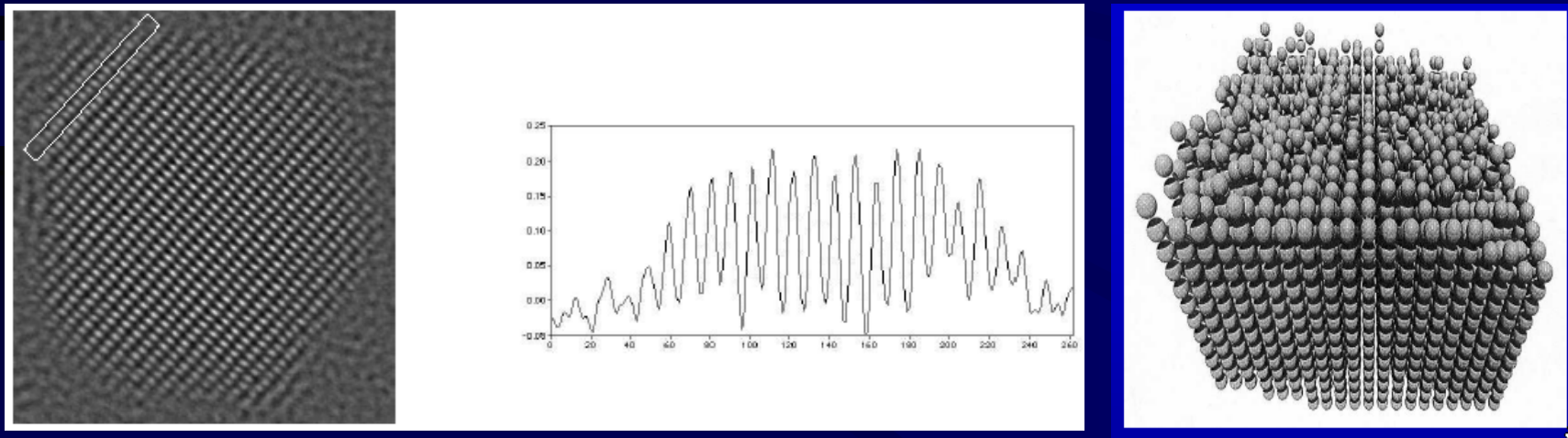
Transmission electron microscopy (TEM): a technique whereby a beam of electrons is transmitted through an ultra thin specimen, interacting with the specimen as it passes through it.



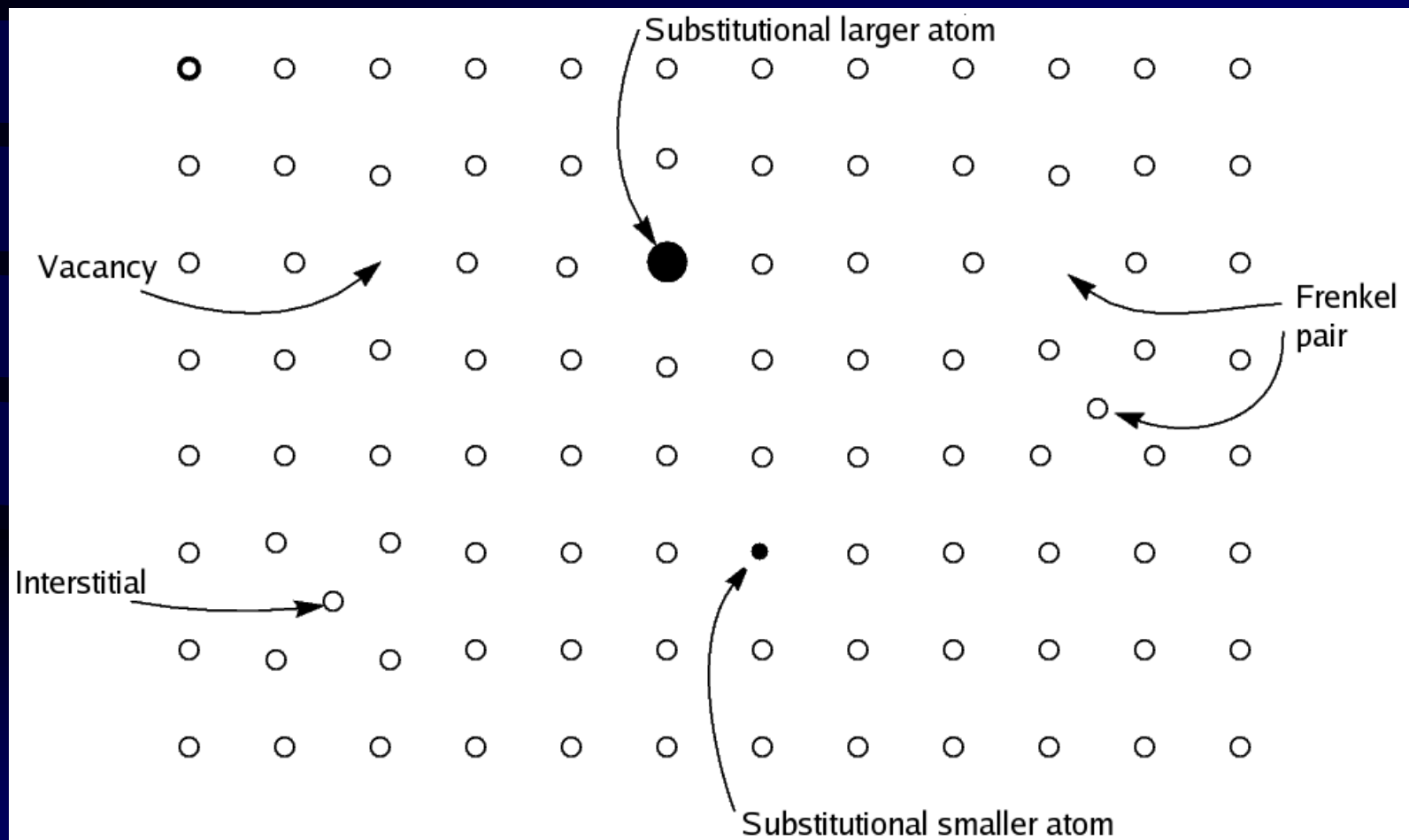
- biological macromolecules are usually composed essentially of ice, protein, and nucleic acid
- the sample may be damaged by the electron beam → few projections

Electron Microscopy II.

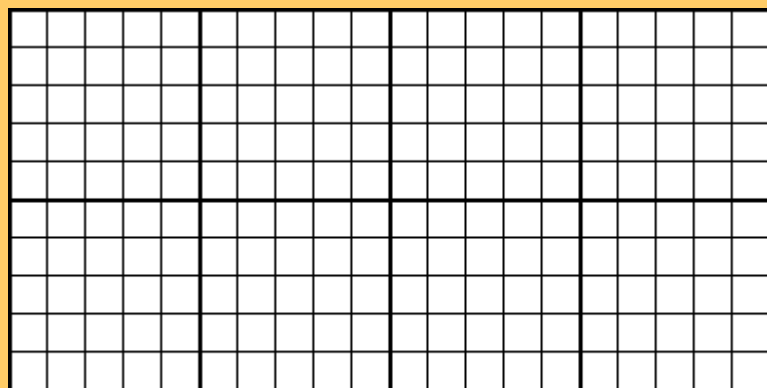
QUANTITEM: a method which provides quantitative information for the number of atoms lying in a single atomic column from HRTEM images



Crystal defects



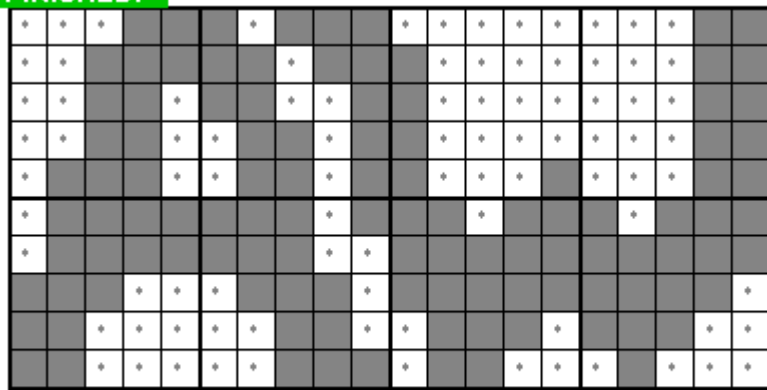
Nonograms



3.3.2.
5.3.2.
2.2.2.2.
2.2.2.2.
3.2.2.1.2.
7.3.3.3.
7.10.
3.3.9.
2.2.3.3.
2.3.2.1.

3 6 7 7 2 3 7 1 2 6 7 5 4 4 4 4 4 4 8 7
2 2 7 3 1

FINISHED!



3.3.2.
5.3.2.
2.2.2.2.
2.2.2.2.
3.2.2.1.2.
7.3.3.3.
7.10.
3.3.9.
2.2.3.3.
2.3.2.1.

3 6 7 7 2 3 7 1 2 6 7 5 4 4 4 4 4 4 8 7
2 2 7 3 1

DIRECT

<http://www.inf.u-szeged.hu/~direct>

Discrete Reconstruction Techniques (DEMO Version) - Mozilla Firefox

File Edit View History Bookmarks Tools Help

http://www.inf.u-szeged.hu/direct/direct.html

Getting Started Latest Headlines

2D Reconstruction Methods

- H-V konvex discrete sets
- H-V konvex discrete sets with absorption
- Reconstruction of circles
- Objects with more gray levels
- Multi-resolution method

Contents

- [What is DIRECT?](#)
- [Generation of data](#)
- [Reconstruction](#)
- [2D Visualization](#)
- [3D Visualization](#)
- [GLView](#)
- [Registration](#)

What is DIRECT?

Discrete REConstruction Techniques (DIRECT) is a toolkit for testing and comparing 2D/3D reconstruction methods of discrete tomography (DT). The toolkit involves generating projections of discrete objects, running reconstruction methods, and visualization of result.

DT deals with reconstruction of cross sections of 2D or 3D discrete objects from their projections. Discrete objects can be divided into two classes: pixel/voxel based images (whose pixels/voxels can take value from a finite set), and parametric images (representing polygons, ellipses, closed 3D regions with triangulated surfaces, etc).

The advantage of this toolkit is the possibility to reach it from Internet. The user needs only a web browser that is able to display [VRML](#) objects. The generation of data, the reconstruction, and the visualization can be parameterized per web-forms.

The process of using this toolkit consists of 3 easy steps. The first step is to generate projections, the second step is the reconstruction, and the last step is the visualization of results (that is also implemented in 3D). When a step is finished the next will be loaded automatically by the browser. If the user has already a file with projections the process can be started at the second step. This feature is available only for [registered](#) users.

The user can download the file containing the result in each step. The DIRECT v1.0 file format can be found [here](#) and DIRECT v2.0 [here](#). The result of visualization is available in [VRML](#) format.

[Back to Contents](#)

Done

start SZTE Informatikai Ta... Discrete Reconstructi... Microsoft Word 15:04

Thank you for your attention!