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Discrete Tomography

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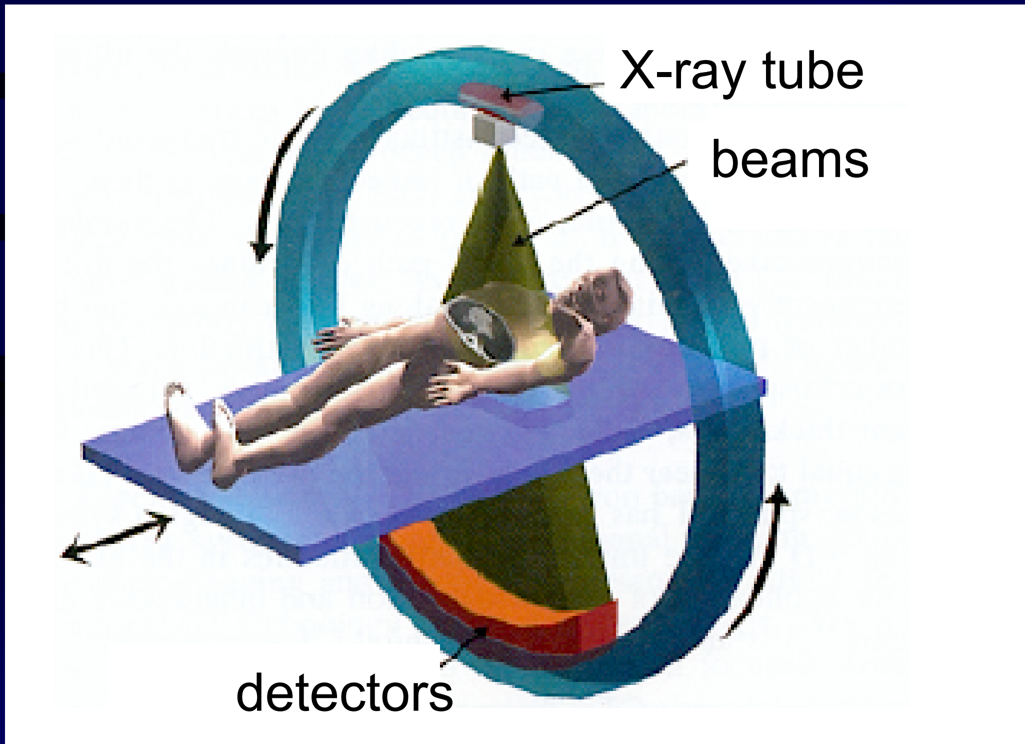
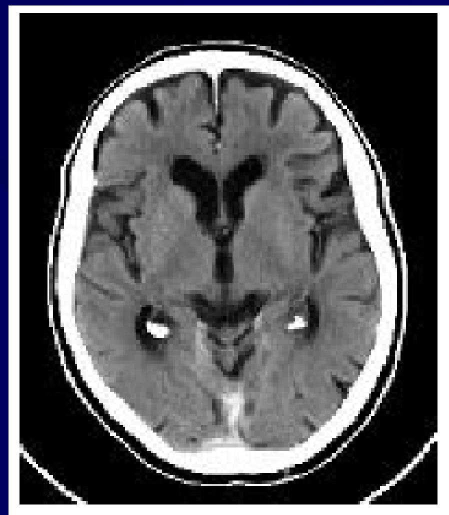
Outline

- Computerized Tomography
- Discrete and Binary Tomography
- Binary Tomography using 2 projections
- Ambiguity and complexity problems
- A priori information
- Reconstruction as optimization
- Applications

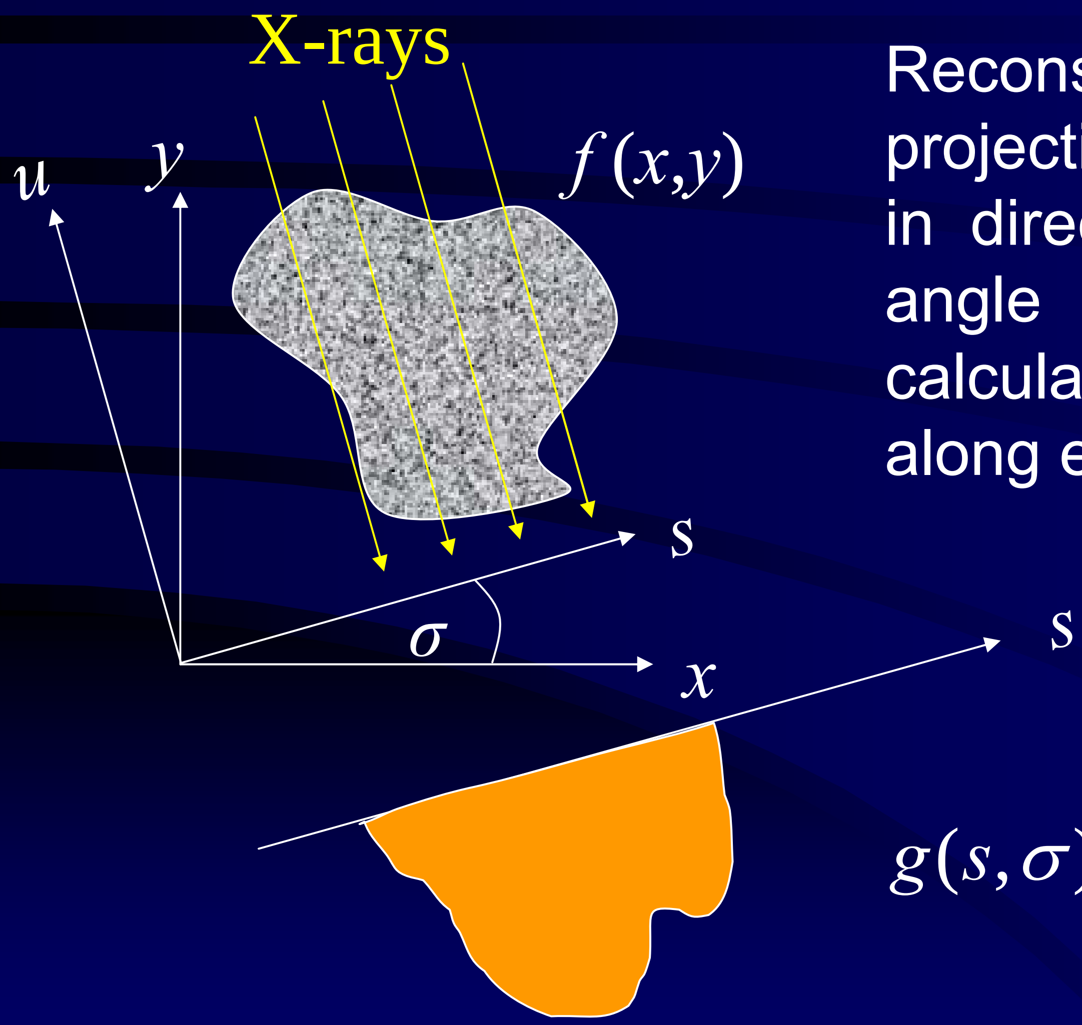
Computerized Tomography

- A technique for imaging the 2D cross-sections of 3D objects (usually human parts)





The Mathematics of CT

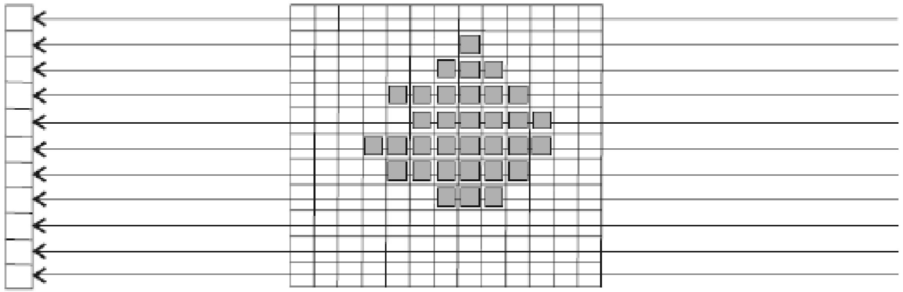


Reconstruct $f\ (x,y)$ from its projections where a projection in direction u (defined by the angle σ) can be obtained by calculating the line integrals along each line parallel to u.

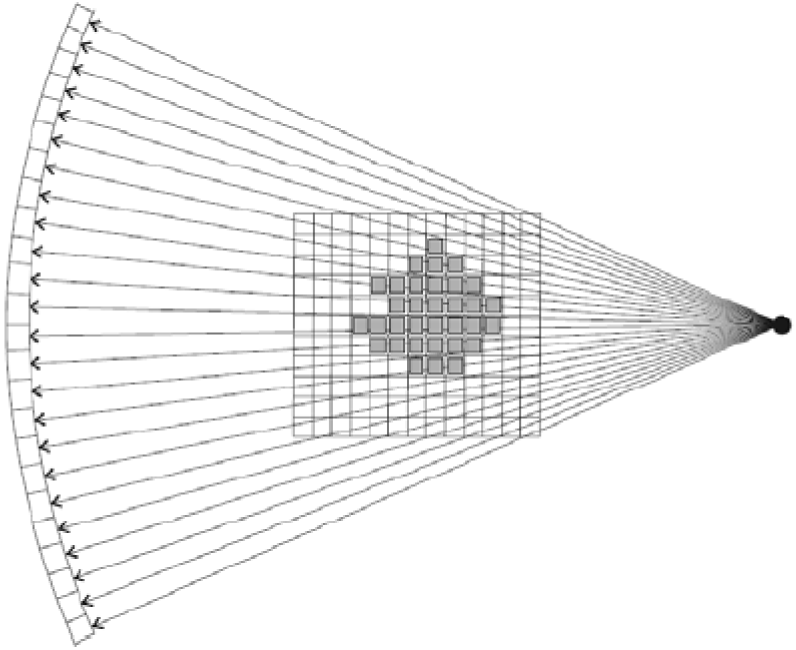
$$g(s,\sigma) = \int_{-\infty}^{\infty} f(x,y)\,du$$

Projection geometries

Parallel

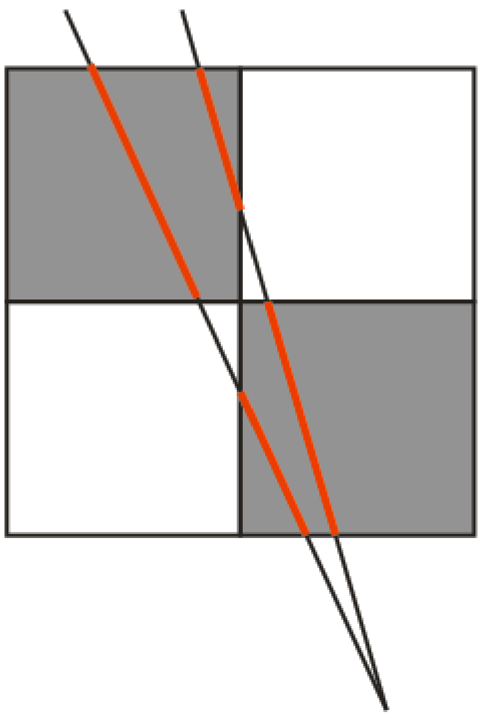


Fan beam

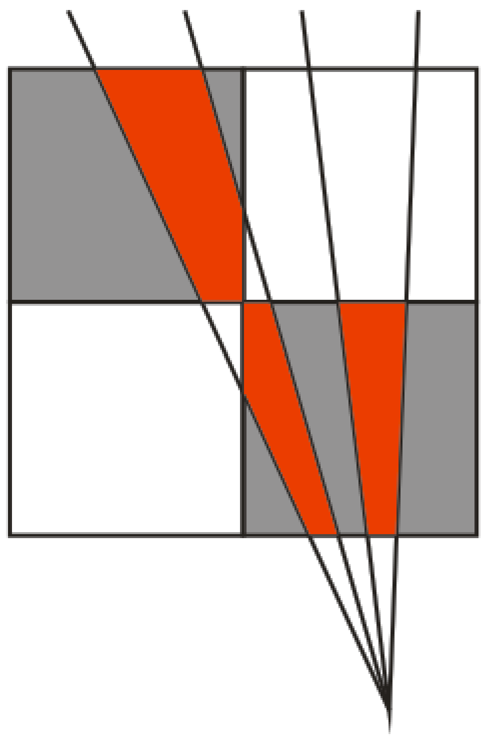


Projections

Line integrals



Area integrals

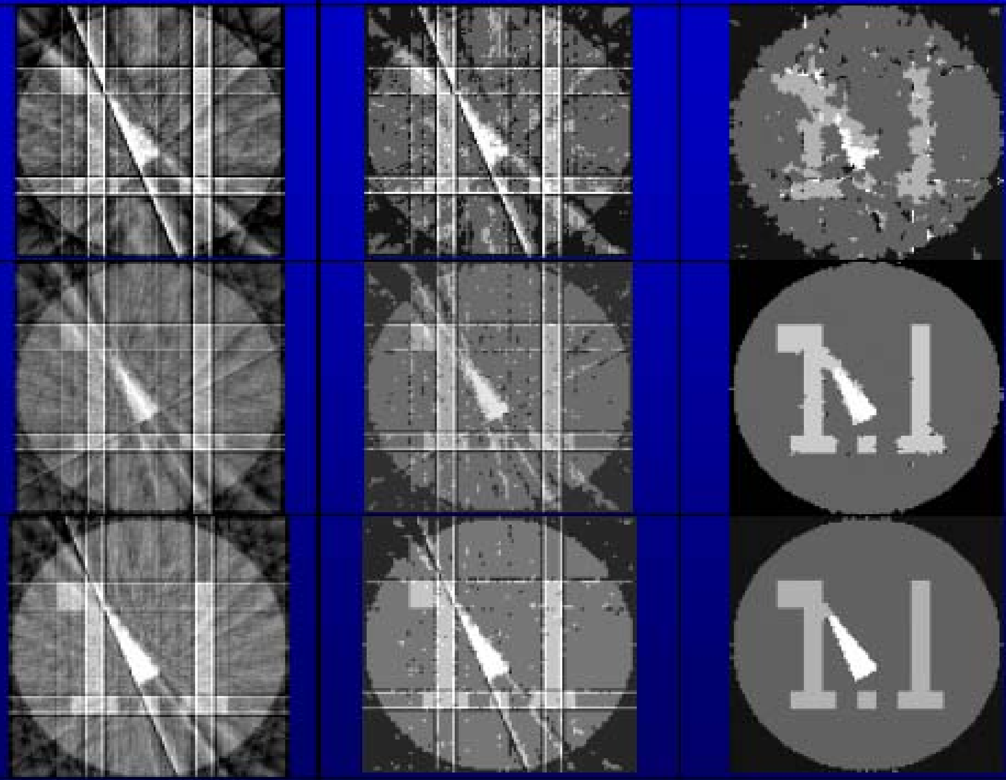


Discrete Tomography

- In CT we need a few hundred projections
  - time consuming
  - expensive
  - may damage the object
- In certain applications the range of the function to be reconstructed is discrete and known $\rightarrow$ DT (only few (2-10) projections are needed)

KNOWING THE DISCRETE RANGE

# projs.	Conv. method	Discretized image	DT method
8			
12			
16			



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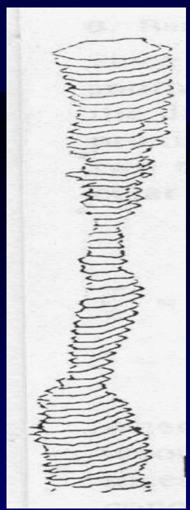
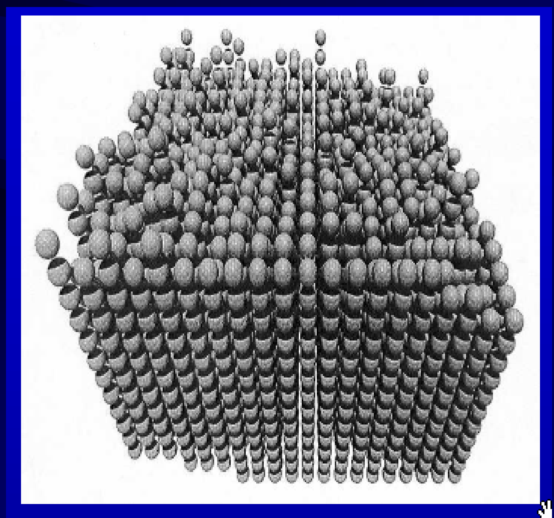
L. Ruskó, A.K., Z. Kiss, L. Rodek, 2003

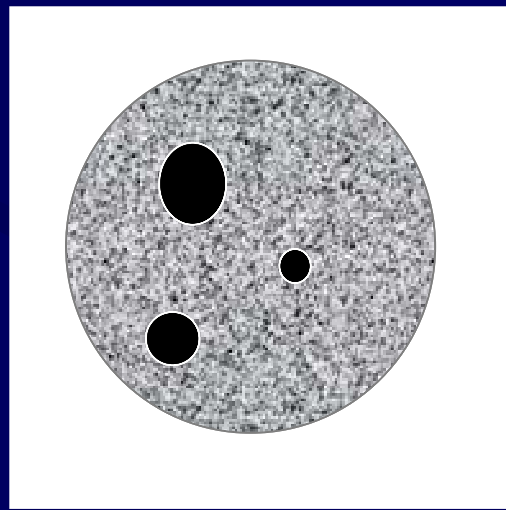
Binary Tomography

the range of the function to be reconstructed is {0,1} (absence or presence of material)

- angiography: parts of human body with X-rays
- electron microscopy: structure of molecules or crystals
- non-destructive testing: obtaining shape information of homogeneous objects

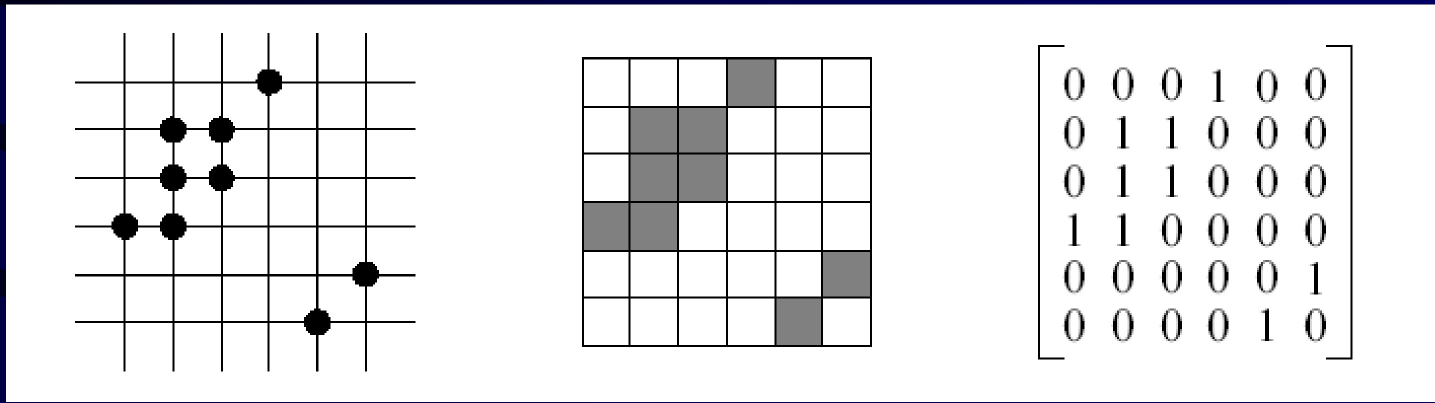




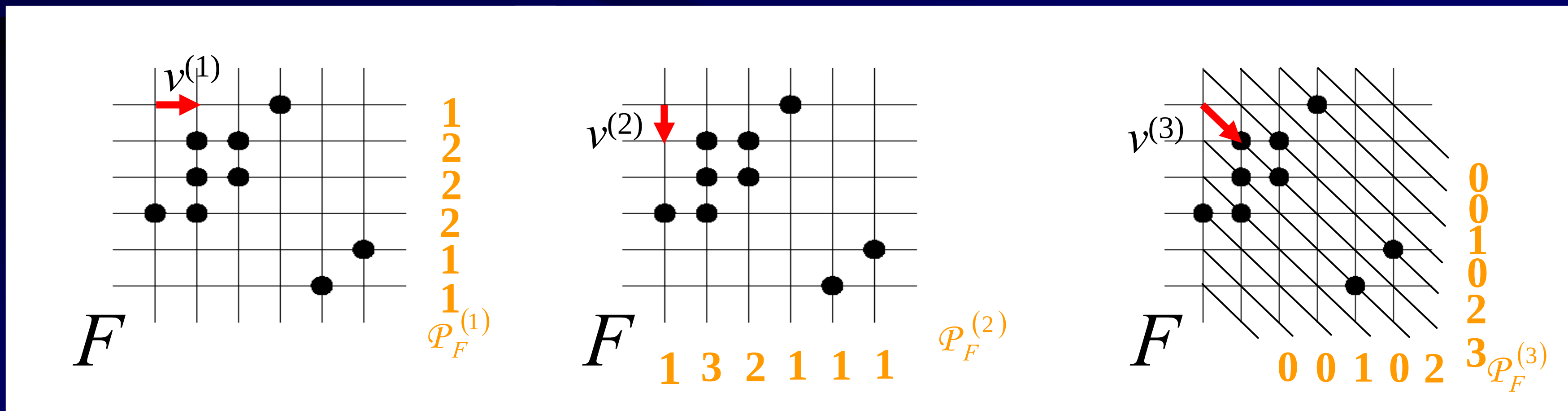


Discrete Sets and Projections

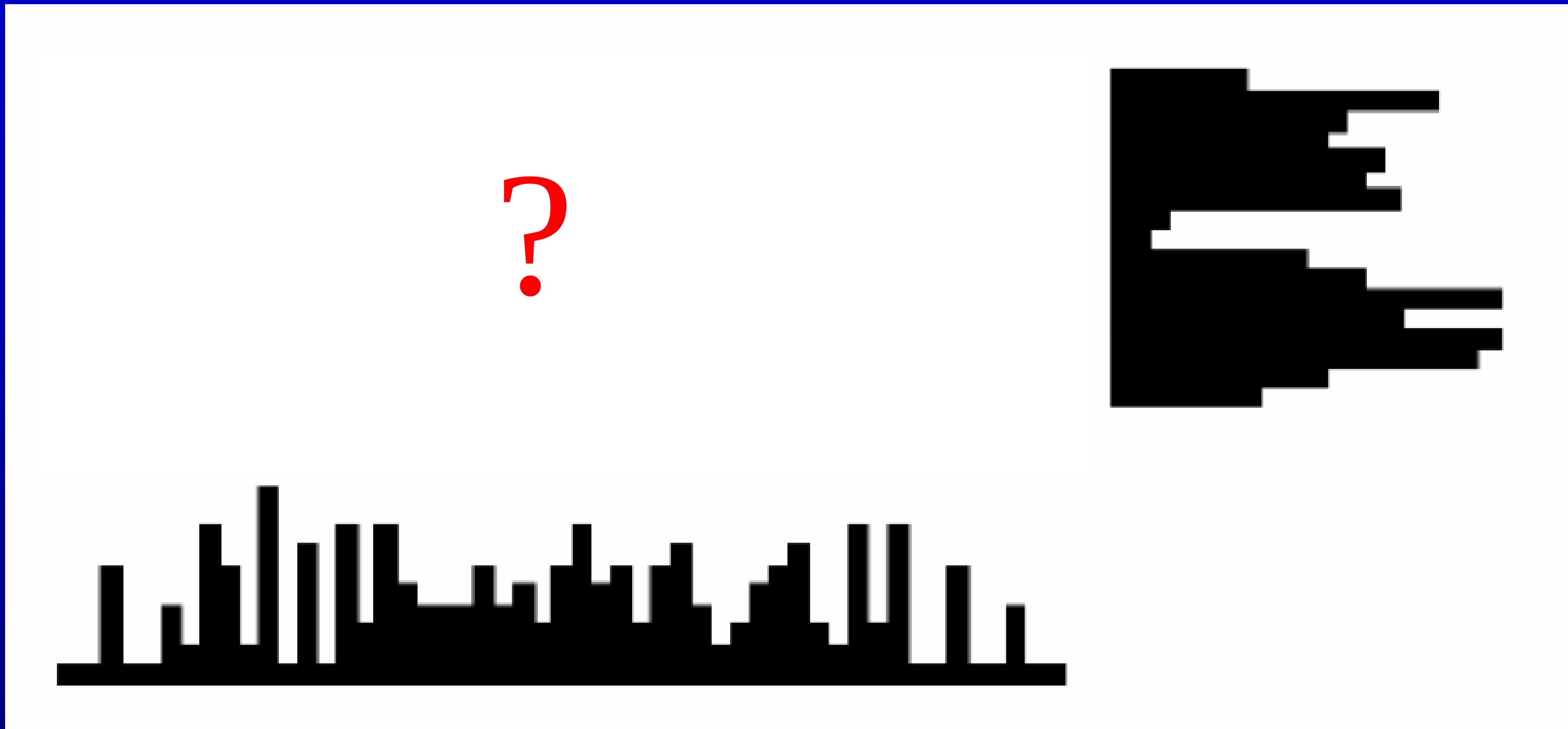
- discrete set: a finite subset of Z^2



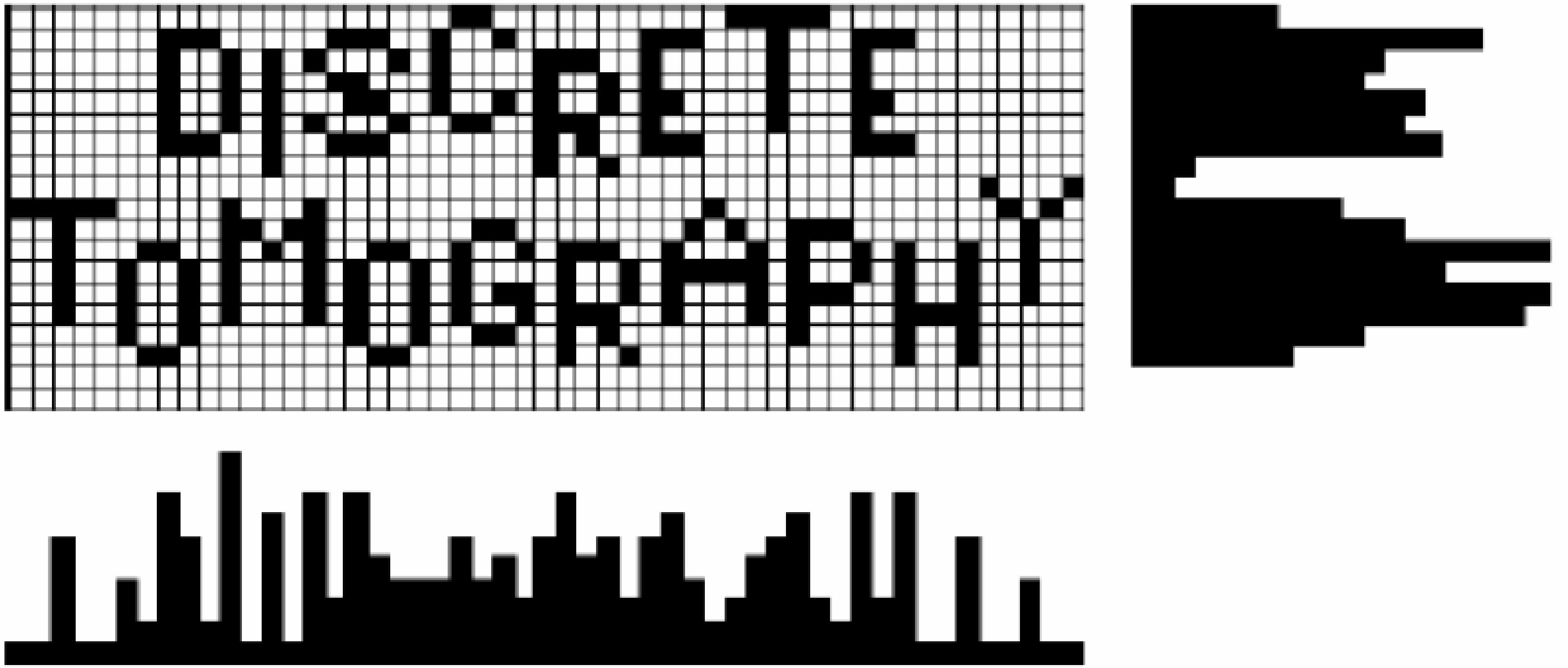
- reconstruct a discrete set from its projections



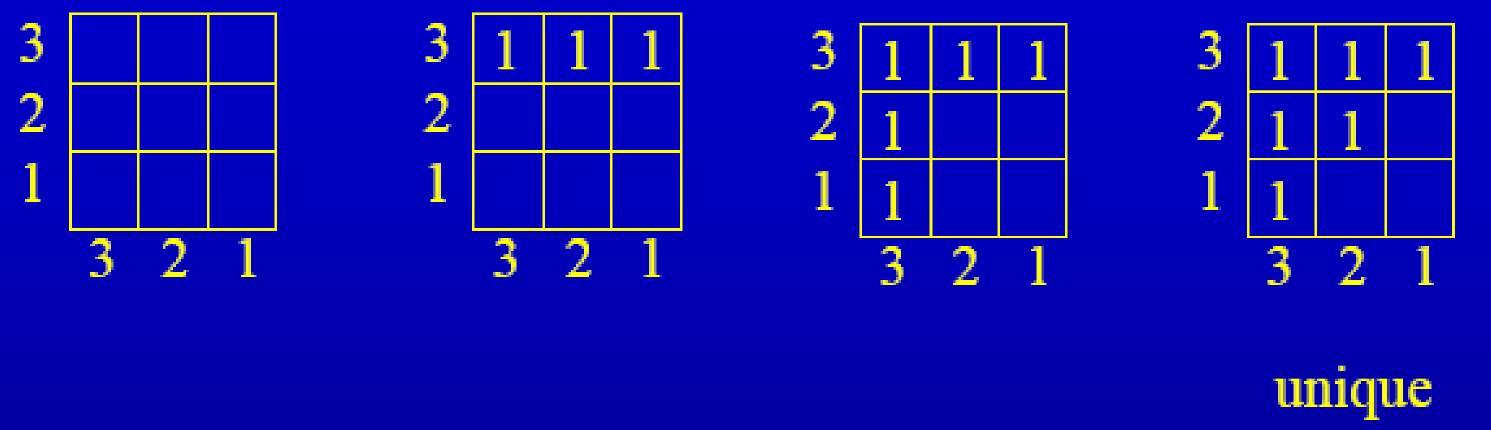
Reconstruction from 2 Projections



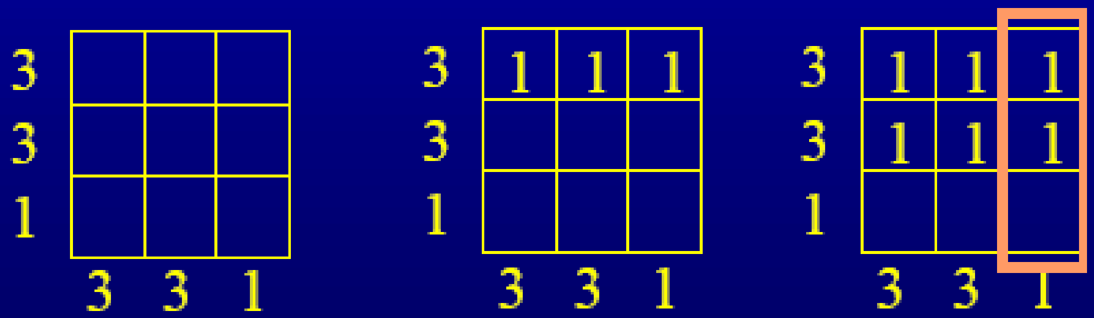
Reconstruction from 2 Projections



Example for Uniqueness

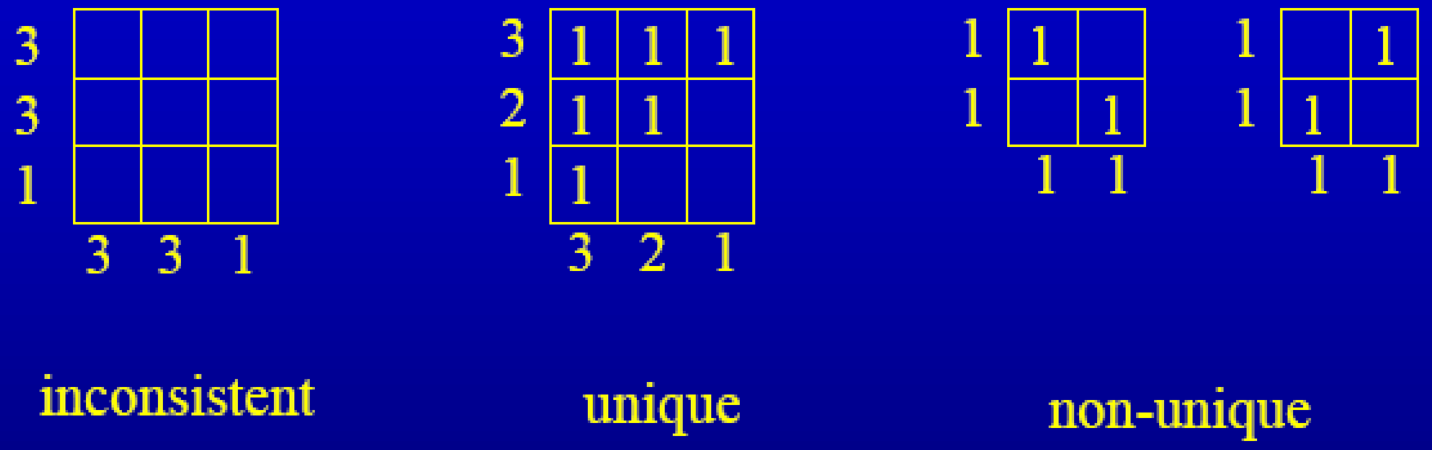


Example for Inconsistency



inconsistent

Classification



Main Problems

Consistency: Does there exist a discrete set with a given set of projections.

Uniqueness: Is a discrete set uniquely determined by a given set of projections.

Reconstruction: Construct a discrete set from its projections.

Reconstruction $\rightarrow$ Consistency

Uniqueness and Switching Components

configuration

1	
	1

	1
1	

2		1	1			
4		1	1	1	1	
3	1	1		1		
4	1	1	1			1
1	1					
	3	4	3	2	1	1

2		1	1			
4		1	1	1		1
3	1	1		1		
4	1	1	1		1	
1	1					
	3	4	3	2	1	1

The presence of a switching component is necessary and sufficient for non-uniqueness

Reconstruction

Ryser, 1957 – from row sums *R* and column sums *S*

Order the elements of *S* in a non-increasing way by $\pi \rightarrow S'$

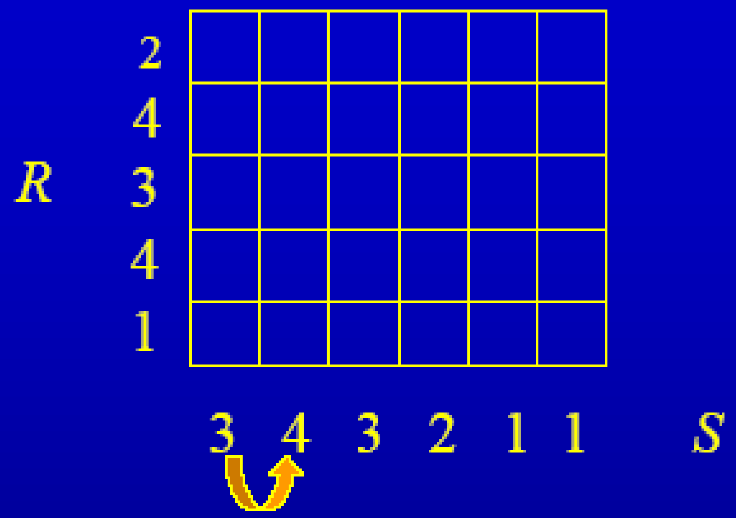
Fill the rows from left to right $\rightarrow B$ (canonical matrix)

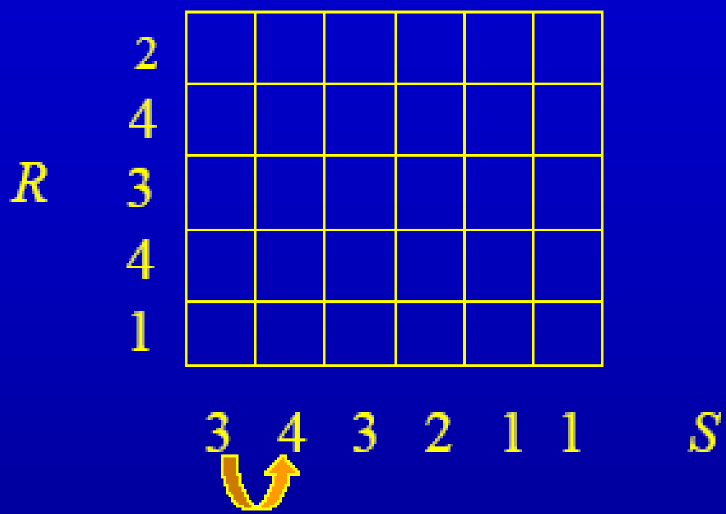
Shift elements from the rightmost columns of *B* to the columns where $S(B) < S'$

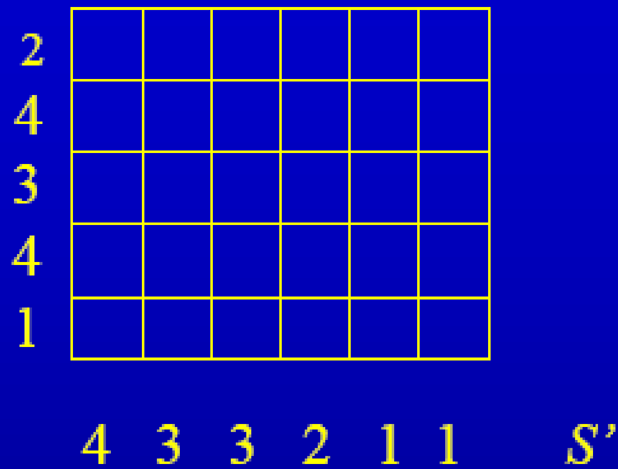
Reorder the colums by applying the inverse of π

Complexity: $O(nm + n \log n)$

R						
2						
4						
3						
4						
1						
S	3	4	3	2	1	1


R
2
4
3
4
1
3 4 3 2 1 1
S


R
2
4
3
4
1
3 4 3 2 1 1
S


2
4
3
4
1
4 3 3 2 1 1
S’

R	2						
	4						
	3						
	4						
	1						
		3	4	3	2	1	1

S

	2						
	4						
	3						
	4						
	1						
		4	3	3	2	1	1

S'

R	2	1	1				
	4	1	1	1	1		
	3	1	1	1			
	4	1	1	1	1		
	1	1					

=B

5	4	3	2	0	0	$S(B)$
4	3	3	2	1	1	S'

R						
2						
4						
3						
4						
1						
S	3	4	3	2	1	1

R						
2						
4						
3						
4						
1						
S'	4	3	3	2	1	1

R						
2	1	1				
4	1	1	1	1		
3	1	1	1			
4	1	1	1	1		
1	1					
S(*B*)	5	4	3	2	0	0
S'	4	3	3	2	1	1

=B

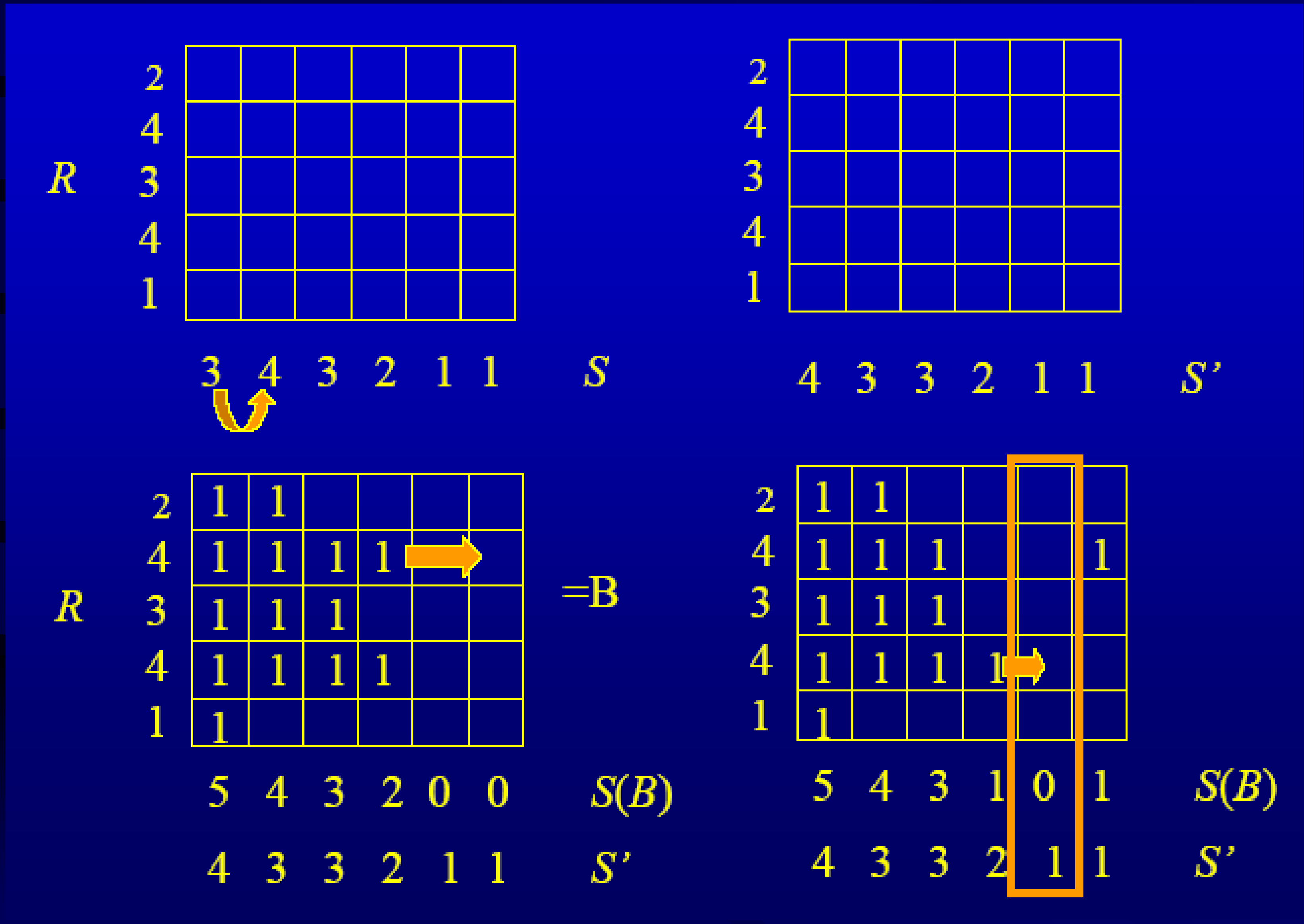
R						
2						
4						
3						
4						
1						
S	3	4	3	2	1	1

R						
2						
4						
3						
4						
1						
S'	4	3	3	2	1	1

R						
2	1	1				
4	1	1	1	1		
3	1	1	1			
4	1	1	1	1		
1	1					
S(B)	5	4	3	2	0	0
S'	4	3	3	2	1	1

=B

2	1	1				
4	1	1	1			1
3	1	1	1			
4	1	1	1	1		
1	1					
S(B)	5	4	3	1	0	1
S'	4	3	3	2	1	1


R
2 4 3 4 1
3 4 3 2 1 1
S
2 4 3 4 1
4 3 3 2 1 1
S’
=B
5 4 3 2 0 0
S(B)
4 3 3 2 1 1
S’
5 4 3 1 0 1
S(B)
4 3 3 2 1 1
S’

R						
2	1	1				
4	1	1	1			1
3	1	1	1			
4	1	1	1		1	
1	1					
$S(B)$	5	4	3	0	1	1
S'	4	3	3	2	1	1

	2	1	1				
	4	1	1	1			1
R	3	1	1	1			
	4	1	1	1		1	
	1	1					
S(*B*)		5	4	3	0	1	1
S'		4	3	3	2	1	1

R

2	1	1				
4	1	1	1 →			1
3	1	1	1 →			
4	1	1	1		1	
1	1					
$S(B)$	5	4	3	0	1	1
S'	4	3	3	2	1	1

2	1	1				
4	1	1		1		1
3	1	1		1		
4	1	1	1		1	
1	1					
$S(B)$	5	4	1	2	1	1
S'	4	3	3	2	1	1

	2	1	1				
	4	1	1	1			1
R	3	1	1	1			
	4	1	1	1		1	
	1	1					
		5	4	3	0	1	1	*S*(*B*)
		4	3	3	2	1	1	*S'*

2	1	1					
4	1	1		1		1	
3	1	1		1			
4	1	1	1		1		
1	1						
	5	4	1	2	1	1	*S*(*B*)
	4	3	3	2	1	1	*S'*

	2	1	1					
	4	1	1		1		1	
R	3	1	1		1			
	4	1	1	1		1		
	1	1						
		5	4	1	2	1	1	*S*(*B*)
		4	3	3	2	1	1	*S'*

R						
2	1	1				
4	1	1	1			1
3	1	1	1			
4	1	1	1		1	
1	1					
$S(B)$	5	4	3	0	1	1
S'	4	3	3	2	1	1

2	1	1				
4	1	1		1		1
3	1	1		1		
4	1	1	1		1	
1	1					
$S(B)$	5	4	1	2	1	1
S'	4	3	3	2	1	1

R						
2	1	1				
4	1	1		1		1
3	1	1		1		
4	1	1	1		1	
1	1					
$S(B)$	5	4	1	2	1	1
S'	4	3	3	2	1	1

2	1		1			
4	1		1	1		1
3	1	1		1		
4	1	1	1		1	
1	1					
$S(B)$	5	2	3	2	1	1
S'	4	3	3	2	1	1

R						
2	1	1				
4	1	1	1 →			1
3	1	1	1 →			
4	1	1	1		1	
1	1					
S(*B*)	5	4	3	0	1	1
S'	4	3	3	2	1	1

2	1	1				
4	1	1		1		1
3	1	1		1		
4	1	1	1		1	
1	1					
S(*B*)	5	4	1	2	1	1
S'	4	3	3	2	1	1

R						
2	1	1 →				
4	1	1 →		1		1
3	1	1		1		
4	1	1	1		1	
1	1					
S(*B*)	5	4	1	2	1	1
S'	4	3	3	2	1	1

2	1 →		1			
4	1		1	1		1
3	1	1		1		
4	1	1	1		1	
1	1					
S(*B*)	5	2	3	2	1	1
S'	4	3	3	2	1	1

	2		1	1			
	4	1		1	1		1
R	3	1	1		1		
	4	1	1	1		1	
	1	1					
S(B)		4	3	3	2	1	1
S'		4	3	3	2	1	1

	2		1	1			
	4	1		1	1		1
R	3	1	1		1		
	4	1	1	1		1	
	1	1					
S(B)		4	3	3	2	1	1
S'		4	3	3	2	1	1

R							
	2		1	1			
	4	1		1	1		1
	3	1	1		1		
	4	1	1	1		1	
	1	1					
		4	3	3	2	1	1	$S(B)$
		4	3	3	2	1	1	S'

2	1		1				
4		1	1	1		1	
3	1	1		1			
4	1	1	1		1		
1		1					
	3	4	3	2	1	1	S

Consistency

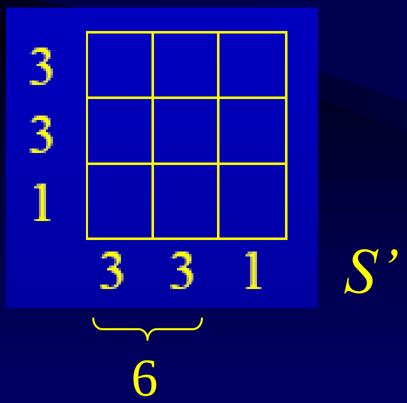
- Necessary condition: compatibility

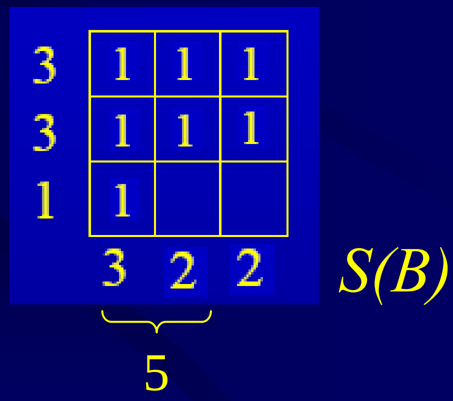
$$\sum_{i=1}^{m} r_i = \sum_{j=1}^{n} s_j$$

$$r_i \le n\ (i = 1,\dots,m),\ \ s_j \le m\ (j = 1,\dots,n)$$

- Gale, Ryser, 1957: there exist a solution iff

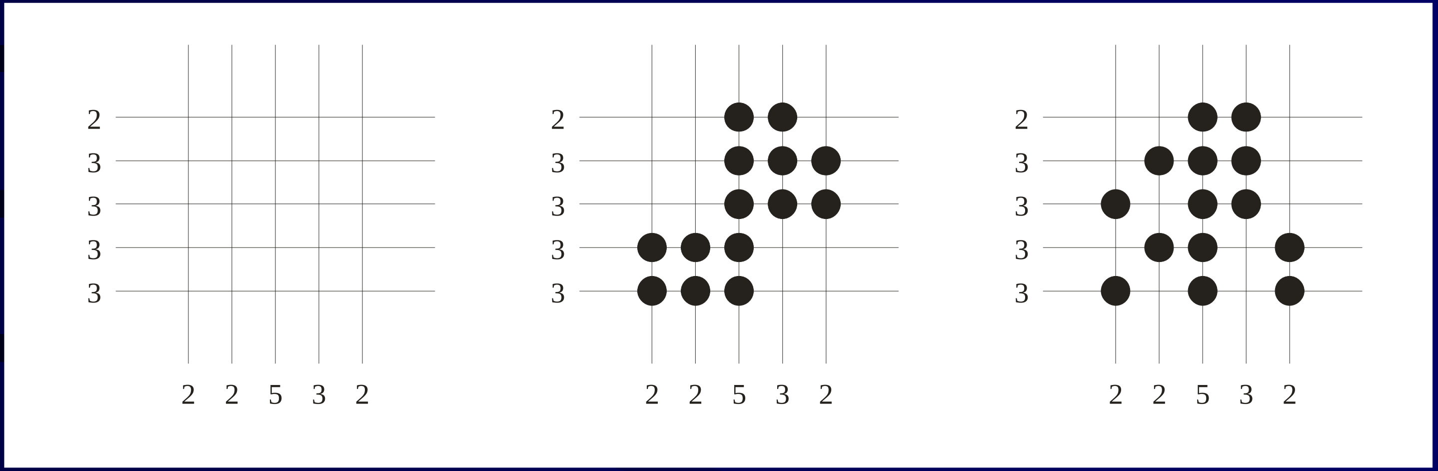
$$\sum_{j=1}^{k} s'_j \le \sum_{j=1}^{k} s(B)_j \quad k = 1,\dots,n$$





Ambiguity

Due to the presence of switching components there can be many solutions with the same two projections



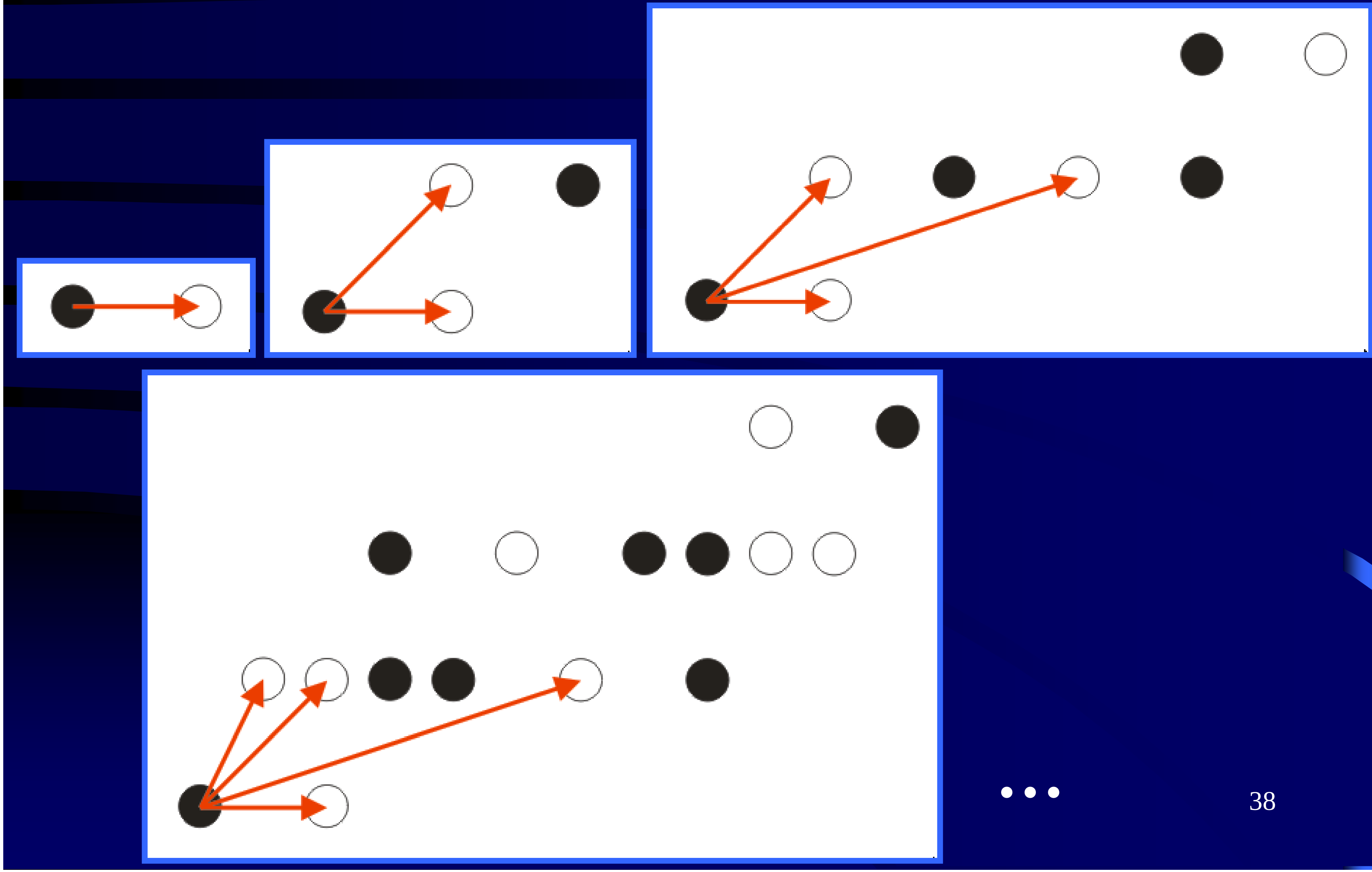
Suggestions:

1. Take further projections along different lattice directions
2. Use a priori information of the set to be reconstructed

Suggestion 1

- In the case of more than 2 projections uniqueness, consistency and reconstruction problems are in general NP-hard – Gardner, Gritzmann 1999
- For an arbitrary number of projections there might be different discrete sets having the same projections

Proof



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Convexity

	1	1			
	1	1	1	1	1
1	1				
1	1	1			
1					

h-convex

	1				
	1		1	1	1
1	1	1			
1	1	1			
1					

v-convex

	1				
	1	1	1	1	1
1	1	1			
1	1	1			
1					

hv-convex

h-convex or *v*-convex: NP-complete - Barcucci et al., 1996

hv-convex: NP-complete - Woeginger, 1996

Connectedness

	1	1			
	1	1	1		1
1	1			1	
1	1	1			
1					

not 4-connected
but 8-connected

	1				
	1		1	1	1
1	1	1	1		
1	1	1			
1					

4-connected

4-connected: NP-complete - Woeginger, 1996

h-convex or *v*-convex, 4-connected: NP-complete - Barcucci *et al.*, 1996

hv-Convex and Connected Sets

hv-convex 8-connected:

	1				
	1	1	1	1	1
1					
1					
1					

- Chrobak, Dürr, 1999

$O(mn \cdot \min\{m^2, n^2\})$

hv-convex 4-connected:

	1				
	1	1	1	1	1
1	1	1			
1	1	1			
1					

- Kuba, 1999

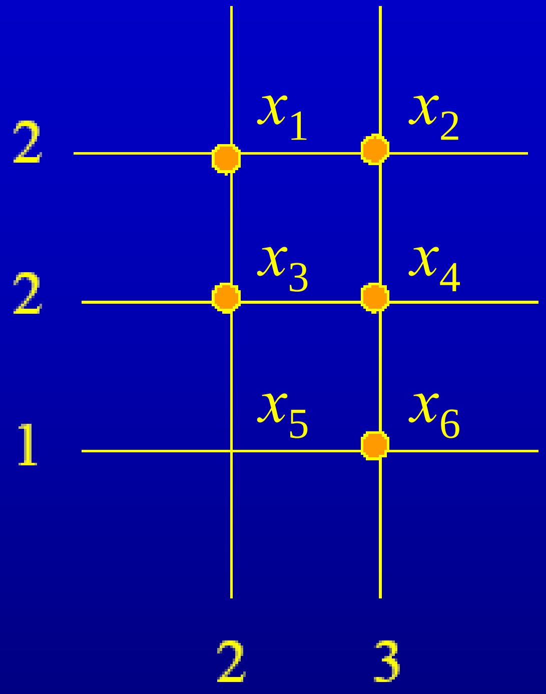
$O(mn \cdot \min\{m^2, n^2\})$

hv-convex 8- but not 4-connected:

- Balázs, Balogh, Kuba, 2005

$O(mn \cdot \min\{m, n\})$

Reconstruction as Optimization



$$\underbrace{\begin{aligned} x_1 + x_2 &= 2 \\ x_3 + x_4 &= 2 \\ x_5 + x_6 &= 1 \\ x_1 + x_3 + x_5 &= 2 \\ x_2 + x_4 + x_6 &= 3 \end{aligned}}_{Px} \quad \Big\} \; b$$

$$P = \begin{pmatrix} 1 & 1 & & & & \\ & & 1 & 1 & & \\ & & & & 1 & 1 \\ 1 & & 1 & & 1 & \\ & 1 & & 1 & & 1 \end{pmatrix} \qquad x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix}$$

Optimization

$$Px = b \qquad x \in \{0,1\}^{m\times n}$$

Problems:

- binary variables
- big system
- underdetermined (#equations << #unknowns)
- inconsistent (if there is noise)

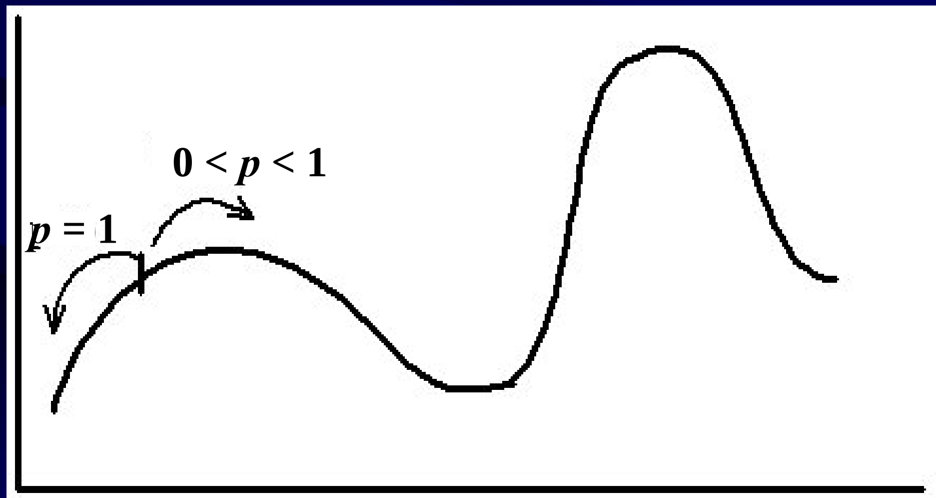
$$x \in \{0,1\}^{m\times n}$$

$$C(x) = \|Px - b\|^2 + g(x) \to \min$$

Term for prior information: convexity, similarity to a model image, etc.

Solving the Optimzation Task

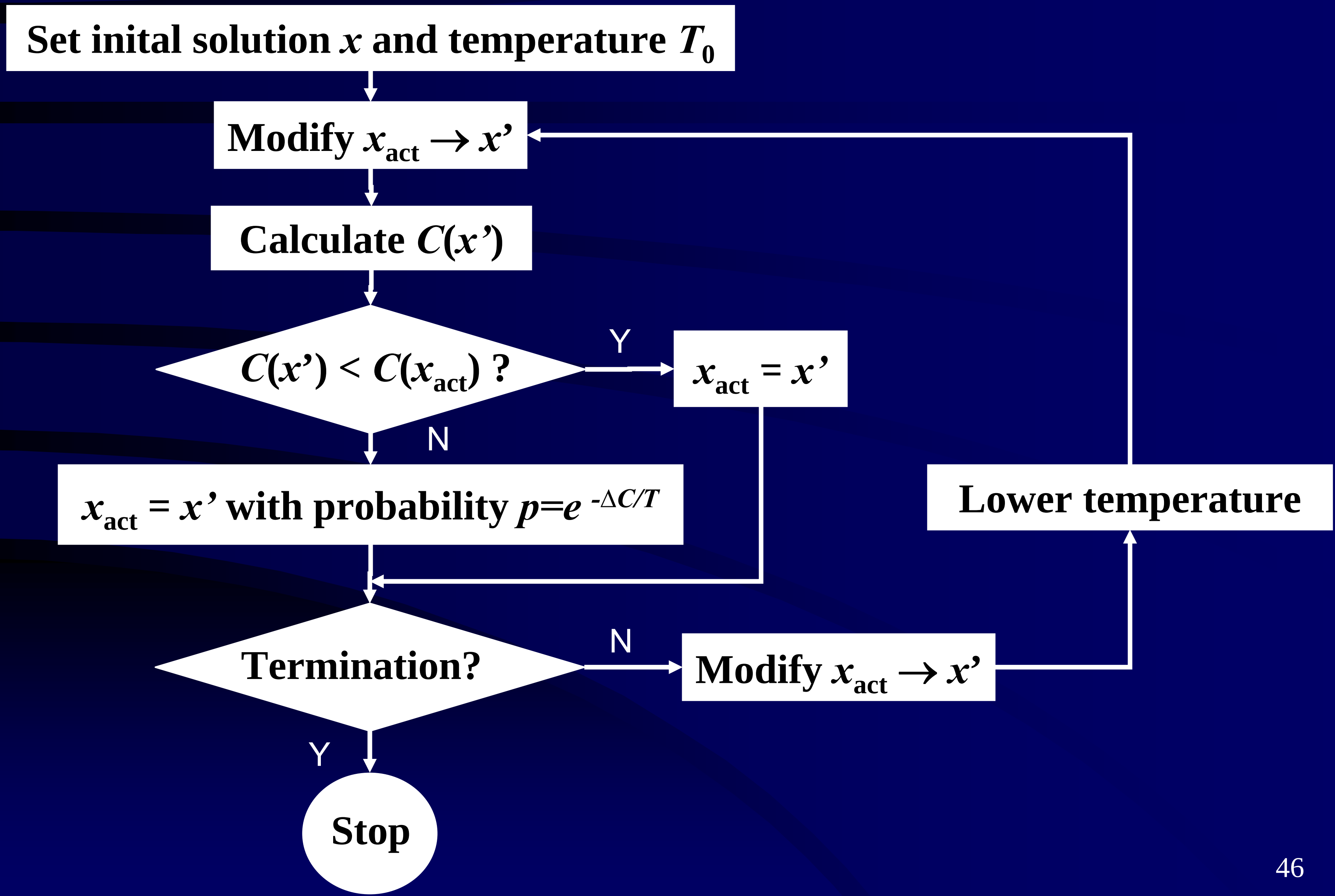
- Problem: Classical hill-climbing algorithms can become trapped in local minima.
- Idea: Allow some changes that increase the objective function.



Simulated Annealing

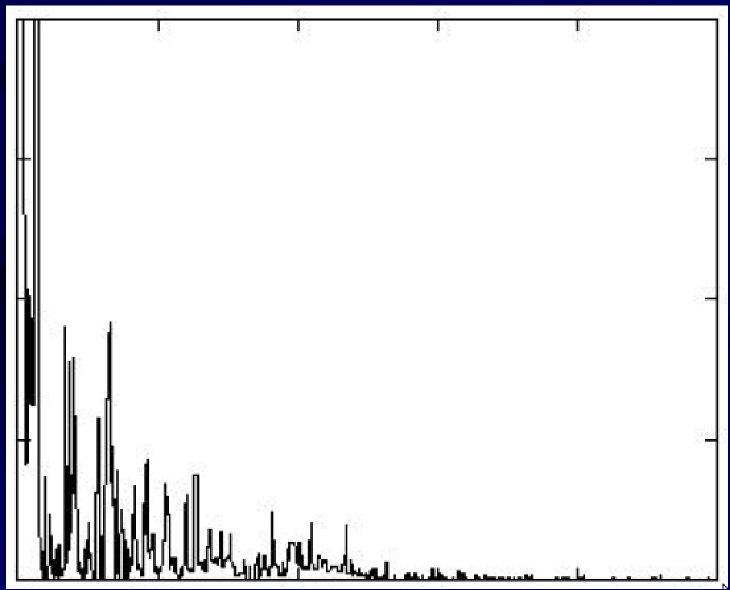
- Annealing: a thermodinamical process in which a metal cools and freezes.
- Due to the thermical noise the energy of the liquid in some cases grows during the annealing .
- By carefully controlling the cooling temperature the fluid freezes into a minimum energy crystalline.
- Simulated annealing: a random-search technique based on the above observation.

Outline of SA



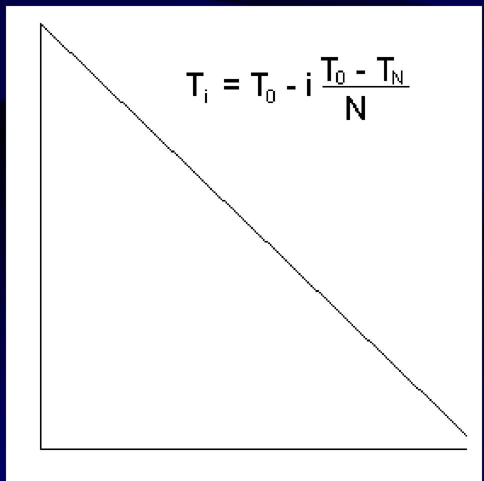
Finding the optimum

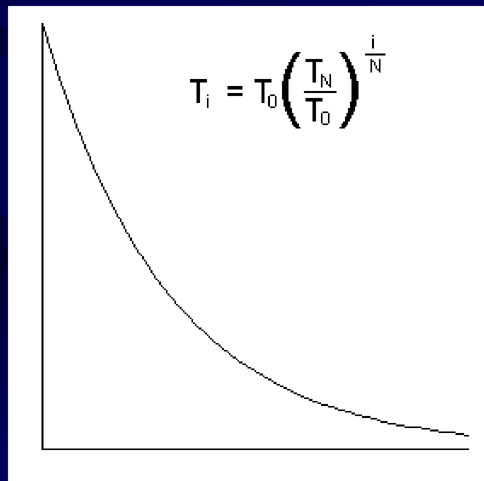
- Tuning the parameters appropriately SA finds the global optimum
- Fine-tuning of the parameters for a given optimization problem can be rather delicate

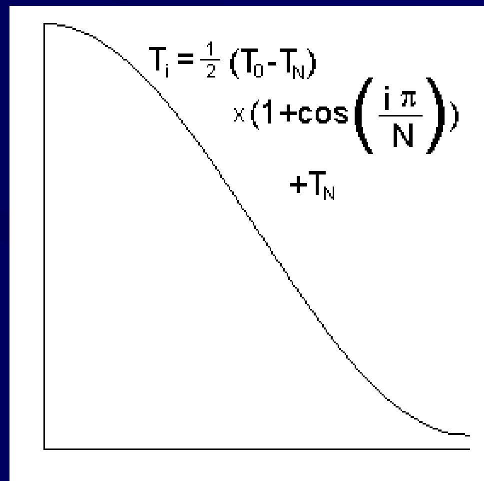


Parameterization of SA

- Initial temperature: T_0
- Stopping criteria: e.g. T_N
- Cooling schedule







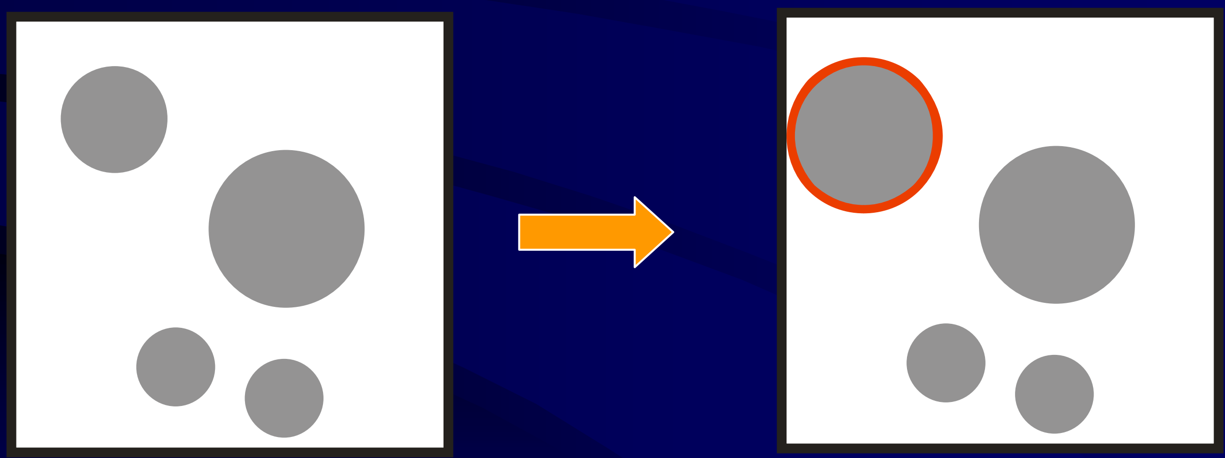
SA in Pixel Based Reconstruction

- A binary matrix describes the binary image
- Randomly invert matrix value(s)

$$\begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 1 & 1 & \boxed{1} \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}$$

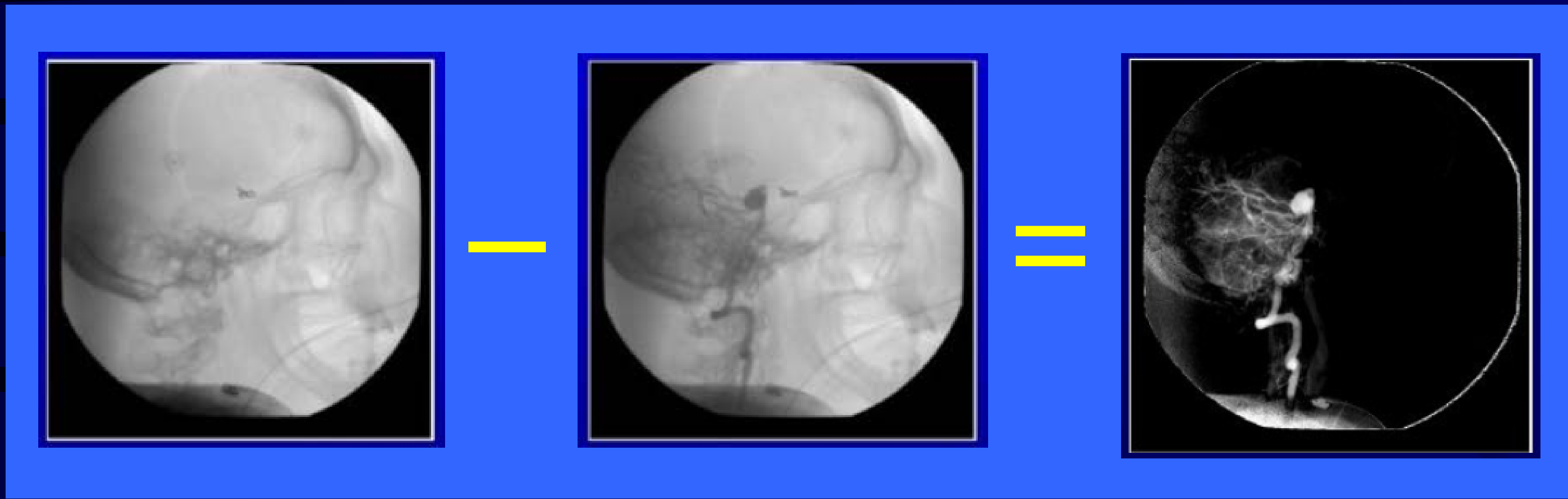
SA in Geometry Based Reconstruction

- The binary image is described by parameters of geometrical objects, e.g. (x,y,r)
- Randomly modify parameter(s) of object(s)

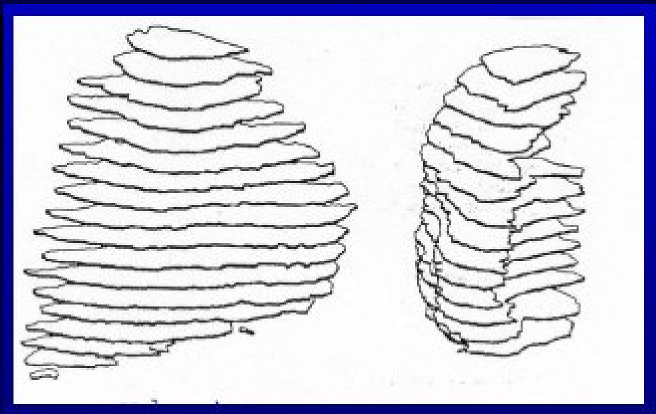


$$[(16,53,17),(44,35,25),(26,13,12),(43,8,12)] \rightarrow [(13,50,23),(44,35,25),(26,13,12),(43,8,12)]$$

Angiography



Heart chambers



Blood vessels

Sources: D.G.W. Onnasch, G.P.M. Prause, T. Schüle

Neighbouring Slices

Slices which are close to each other in space or time are similar

previous slice

	1	1	1		
	1	1	1	1	
		1	1		

cost matrix

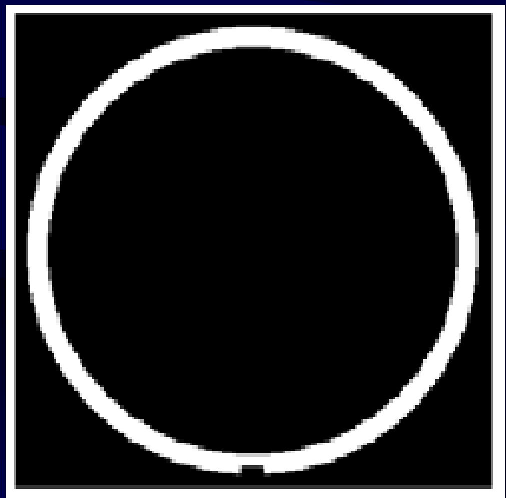
8	7	6	7	8	9
7	4	3	4	5	8
7	4	2	2	4	7
9	8	4	4	5	8
9	9	7	7	8	9

$$x \in \{0,1\}^{m \times n}$$

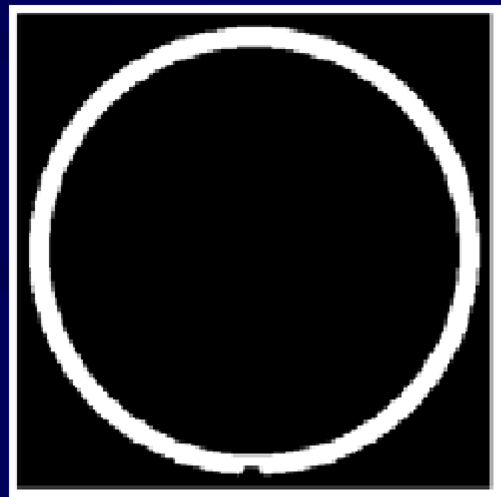
$$C(x) = \|Px - b\|^2 + \sum_{i,j} c_{ij} x_{ij} \rightarrow \min$$

Non-destructive testing

- Pipe corrosion and deposit study
  - 32 fan beam projections



no noise

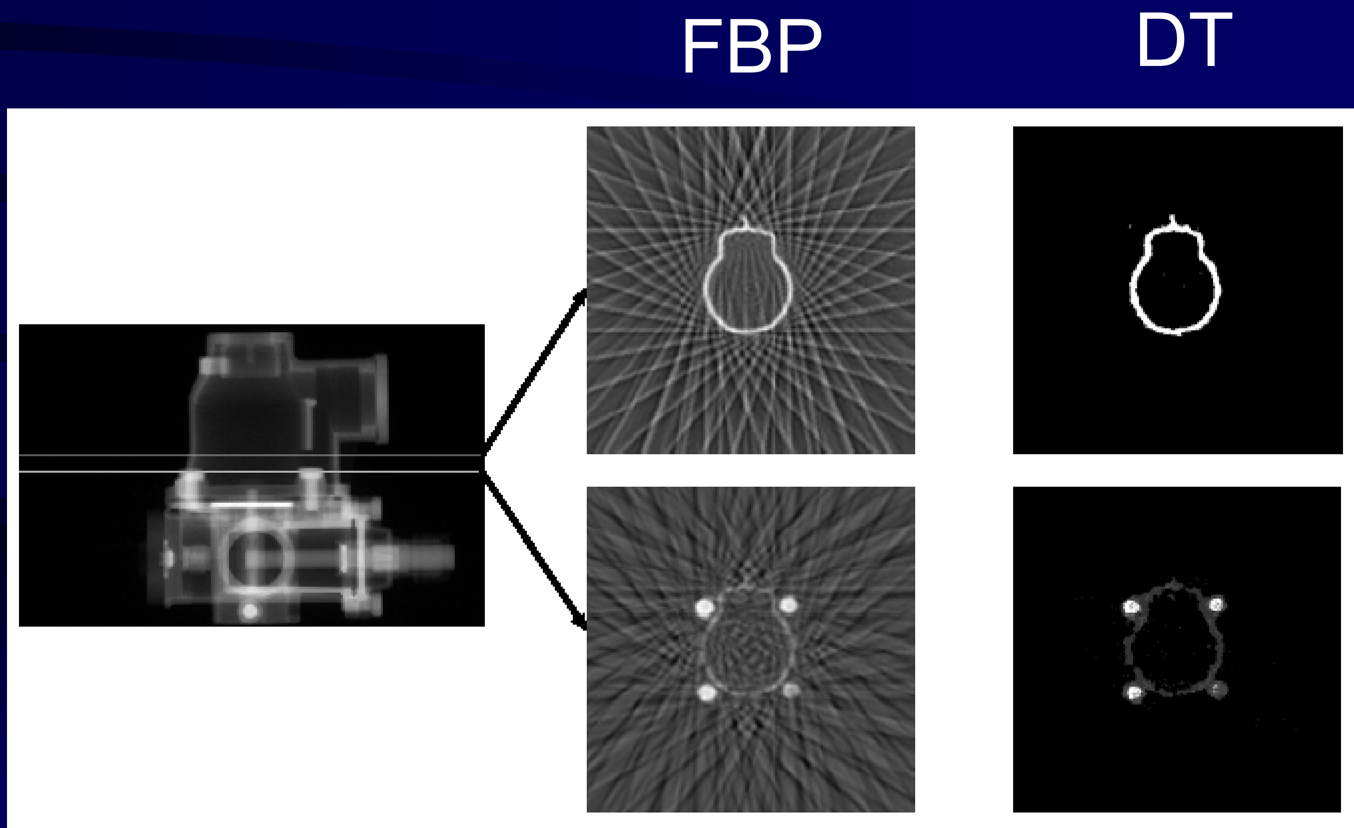


10 % Gaussian noise

Source: A. Nagy

Neutron Tomography I.

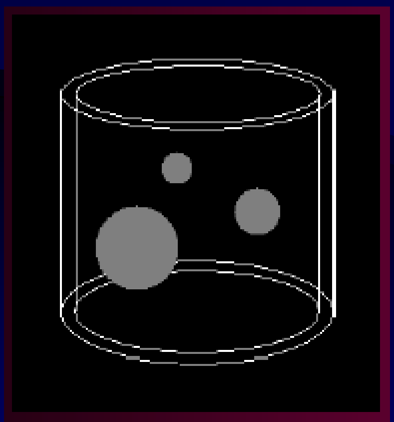
- Gas pressure controller
  - 18 projections, pixel based



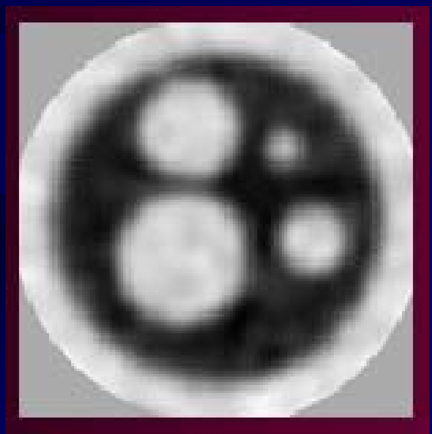
Source: A. Nagy

Neutron Tomography II.

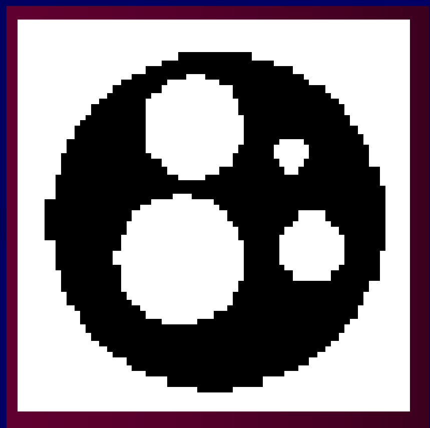
- Reconstruction of disks (air bubbles)
  - 4 projections, geometry based



FBP 60 proj.



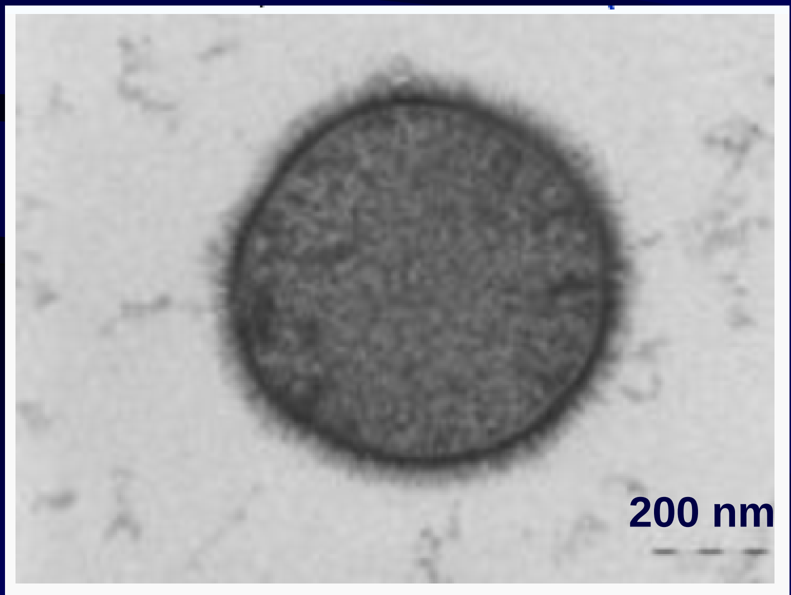
DT 4 proj.



Source: L. Rodek

Electron Microscopy I.

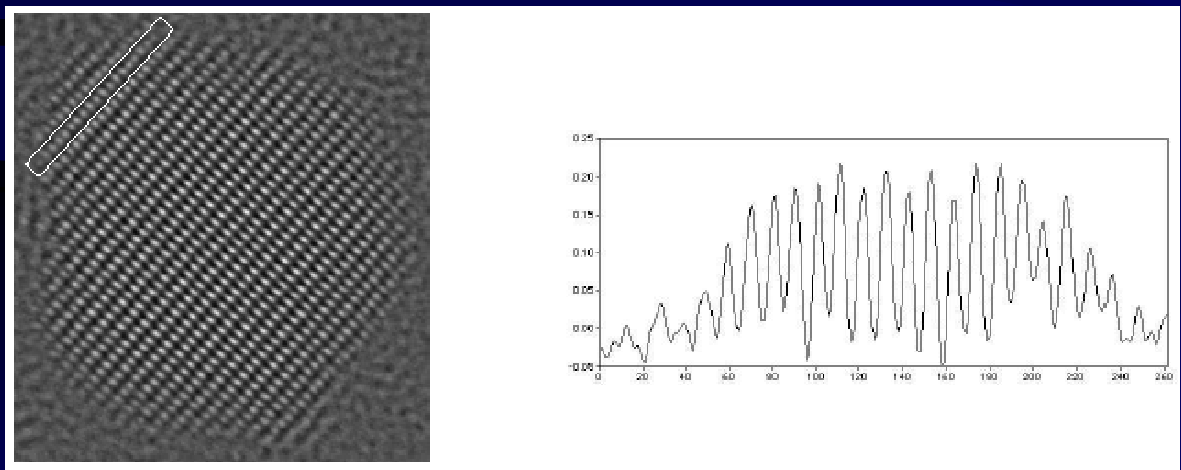
Transmission electron microscopy (TEM): a technique whereby a beam of electrons is transmitted through an ultra thin specimen, interacting with the specimen as it passes through it.

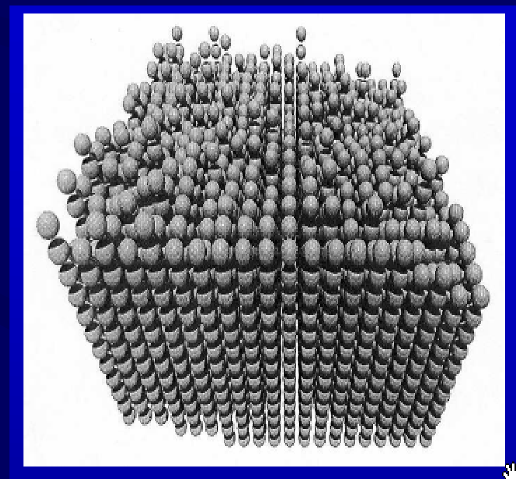


- biological macromolecules are usually composed esentially of ice, protein, and nucleic acid
- the sample may be damaged by the electron beam → few projections

Electron Microscopy II.

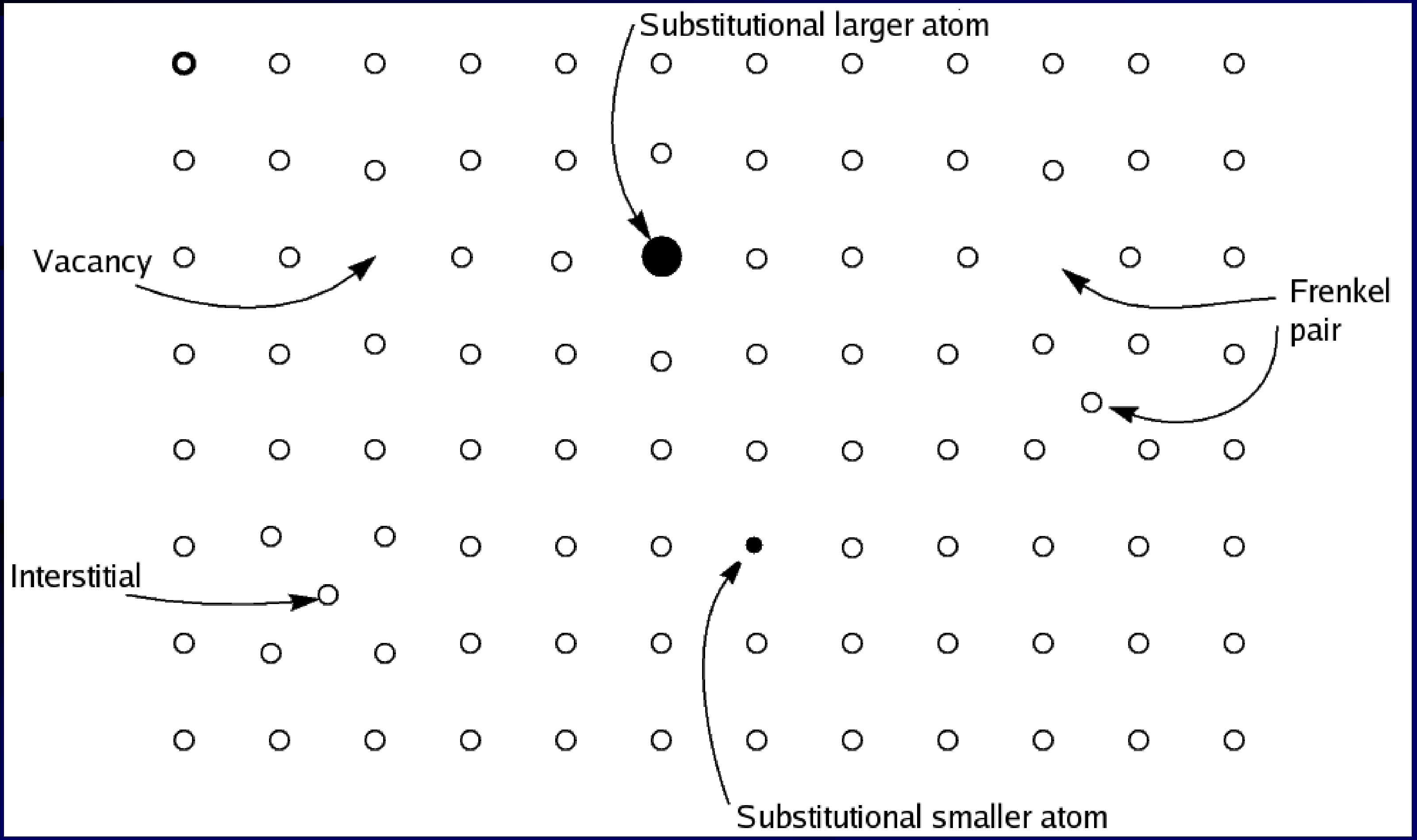
QUANTITEM: a method which provides quantitative information for the number of atoms lying in a single atomic column from HRTEM images



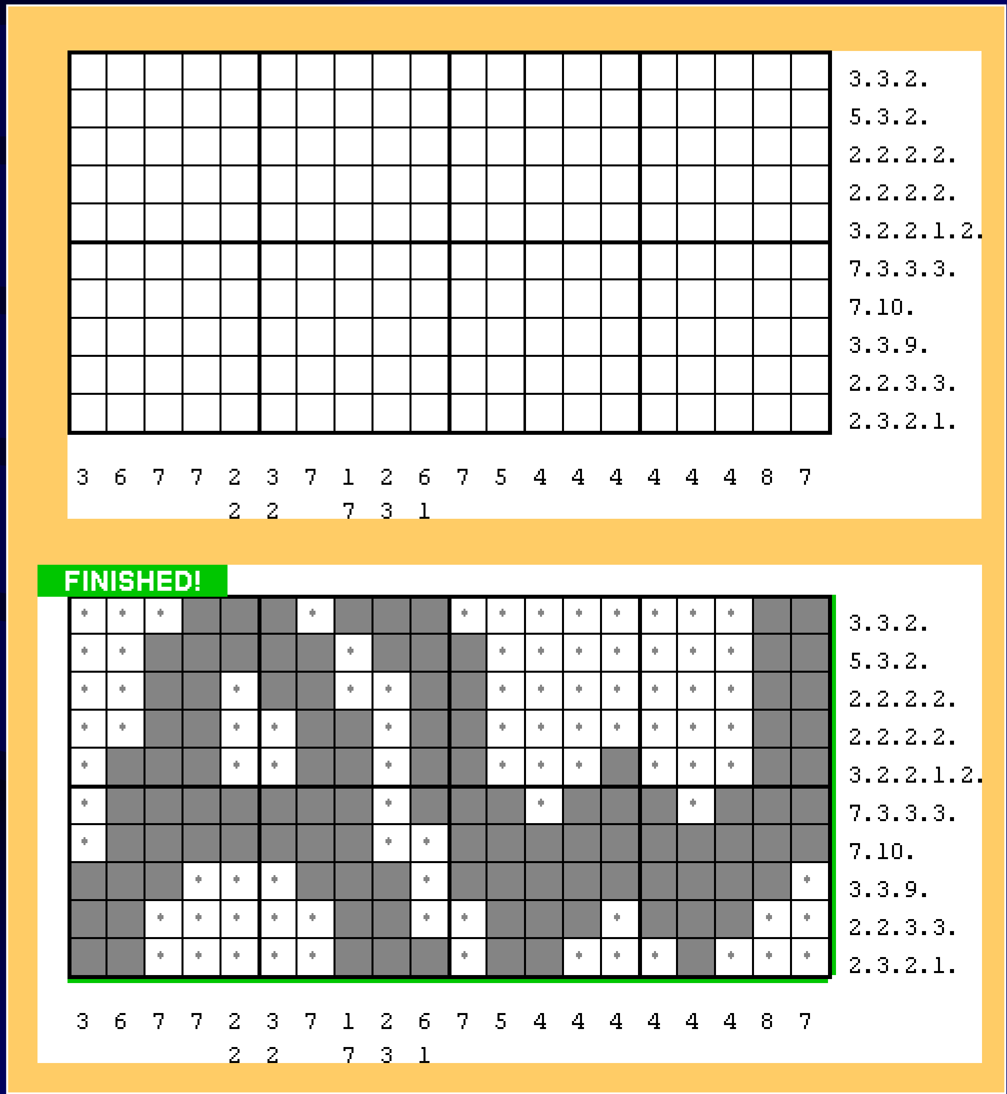


Source: Batenburg, Palenstijn

Crystal defects



Nonograms



Source: Internet

DIRECT
http://www.inf.u-szeged.hu/~direct

Thank you for your attention!