# Computed Tomography

## From X-ray Physics to Applications

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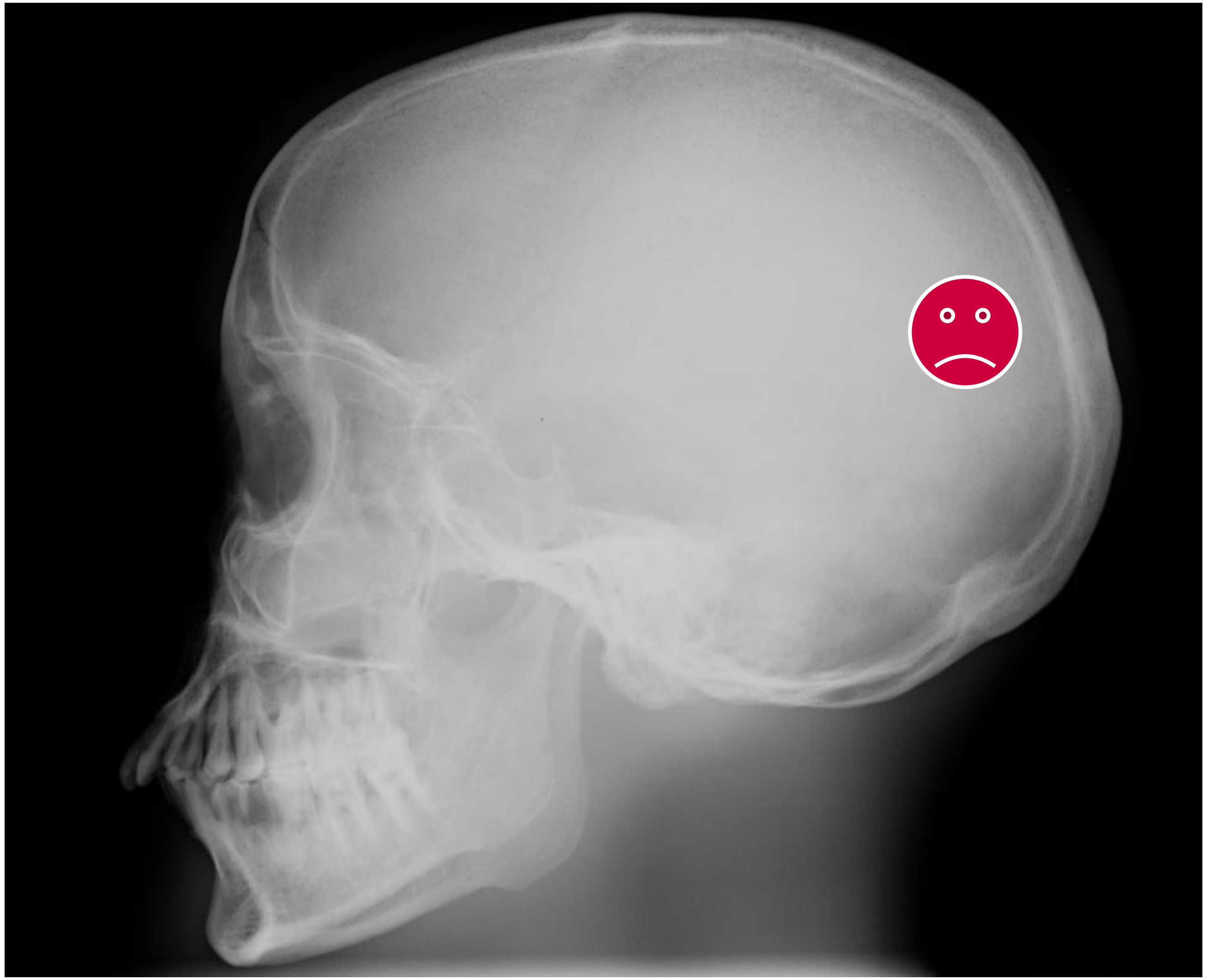
Summer School Szeged 12/07/2016

# Overview

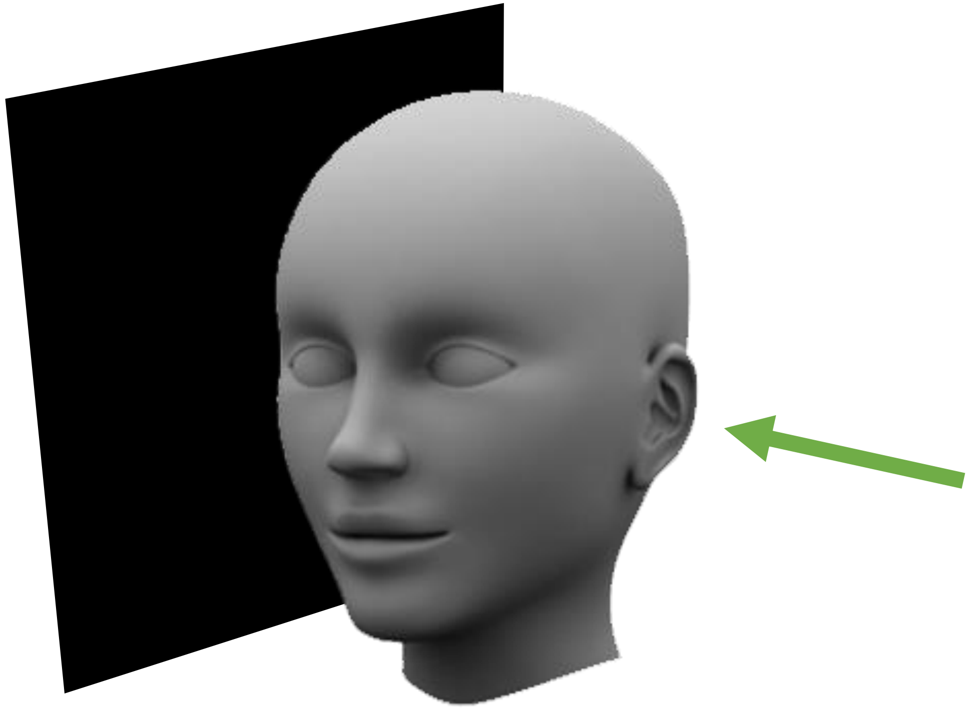


- **Computed Tomography**
- Some Applications
- A Brief Introduction to X-ray Physics
- A Brief Introduction to Reconstruction Mathematics
- When Physics and Mathematics colide

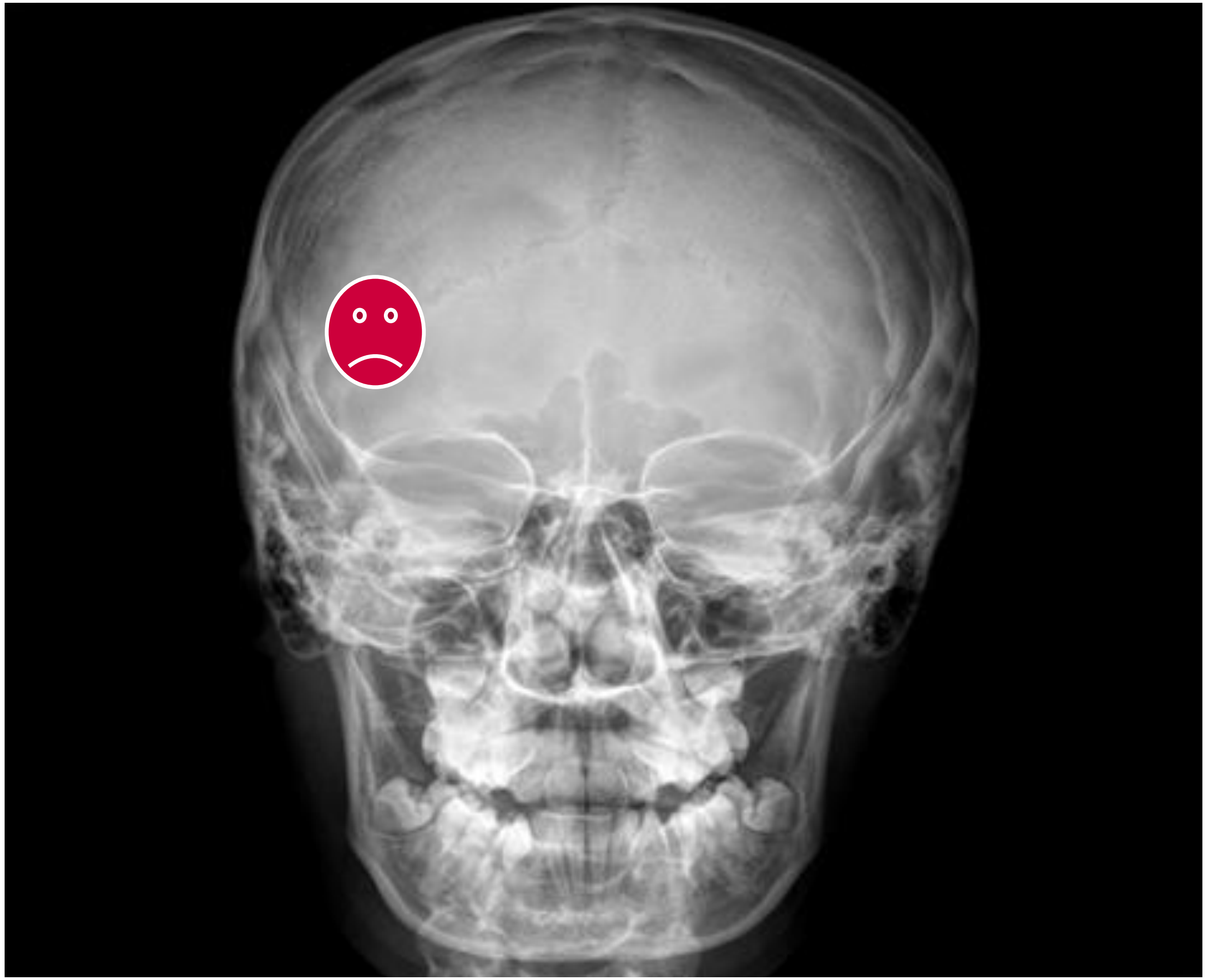
# What is Computed Tomography



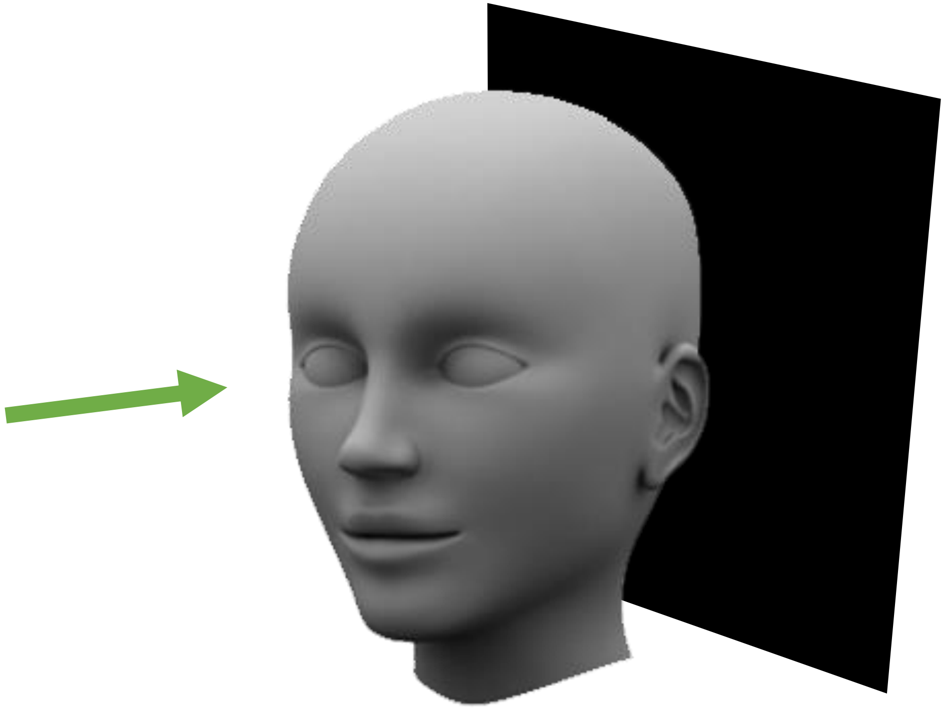
X-ray imaging



# What is Computed Tomography



X-ray imaging



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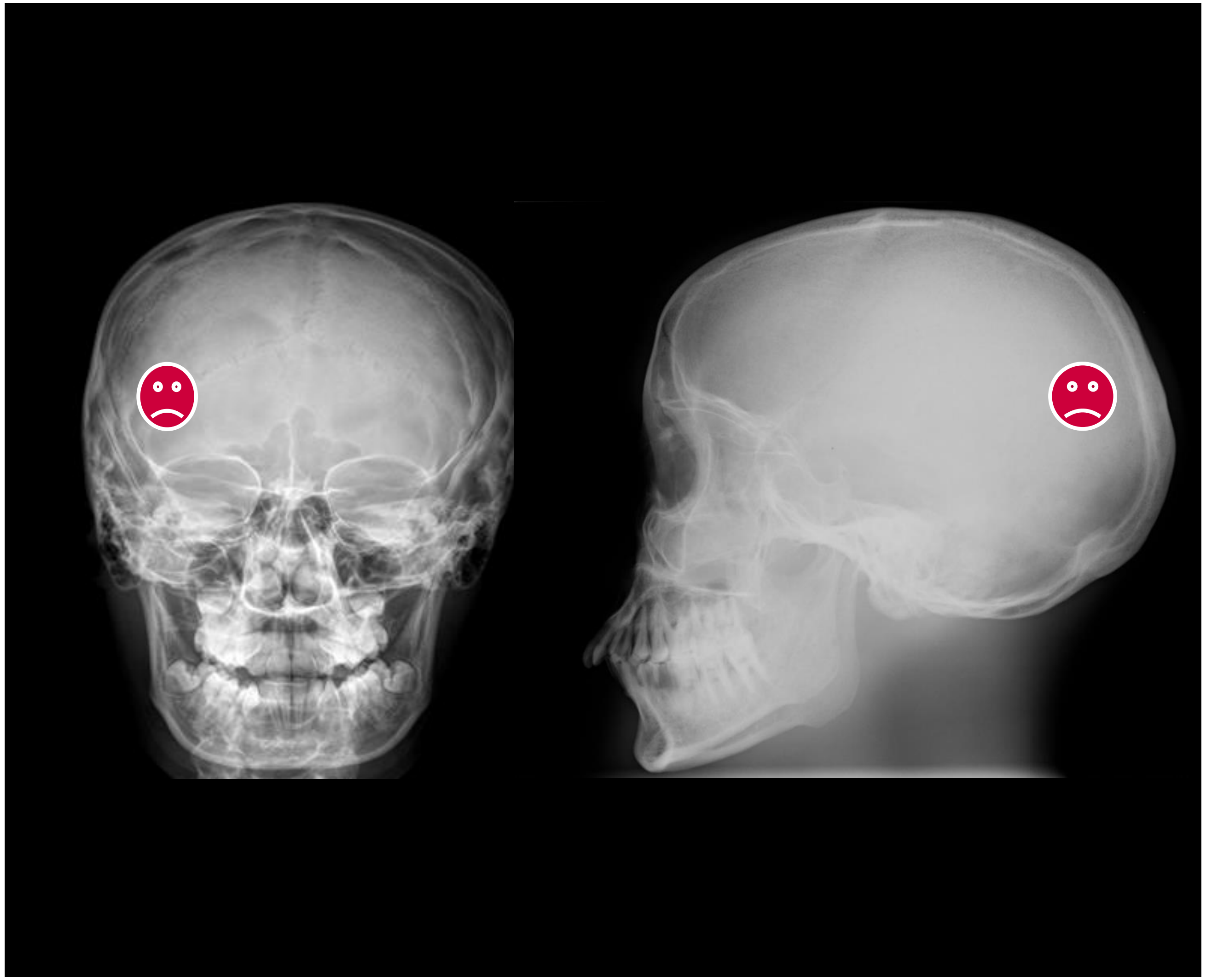
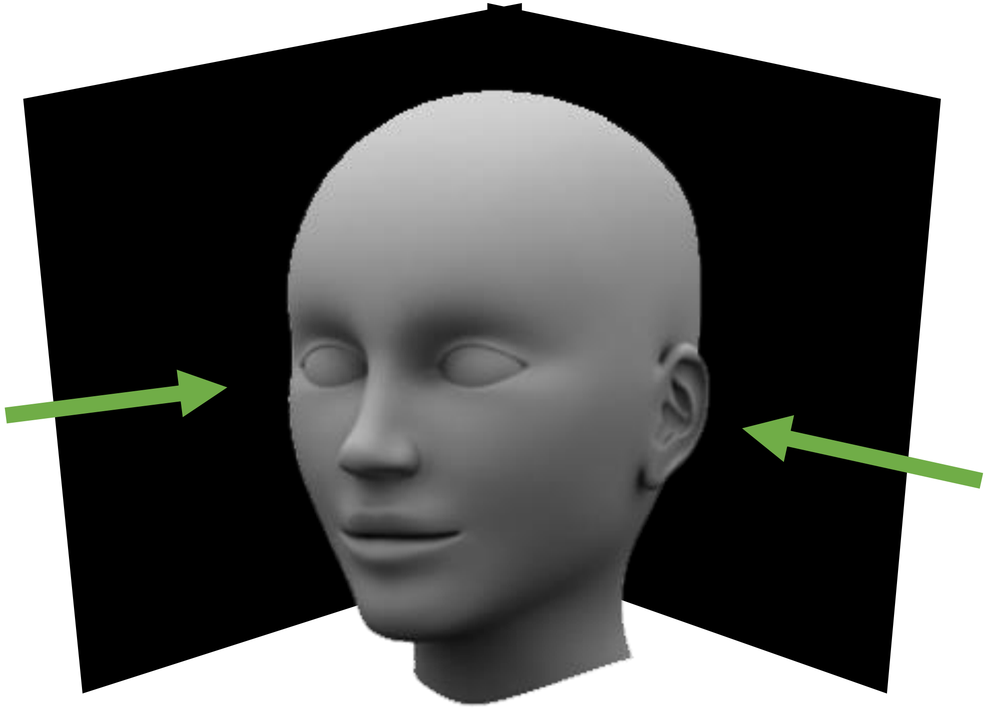
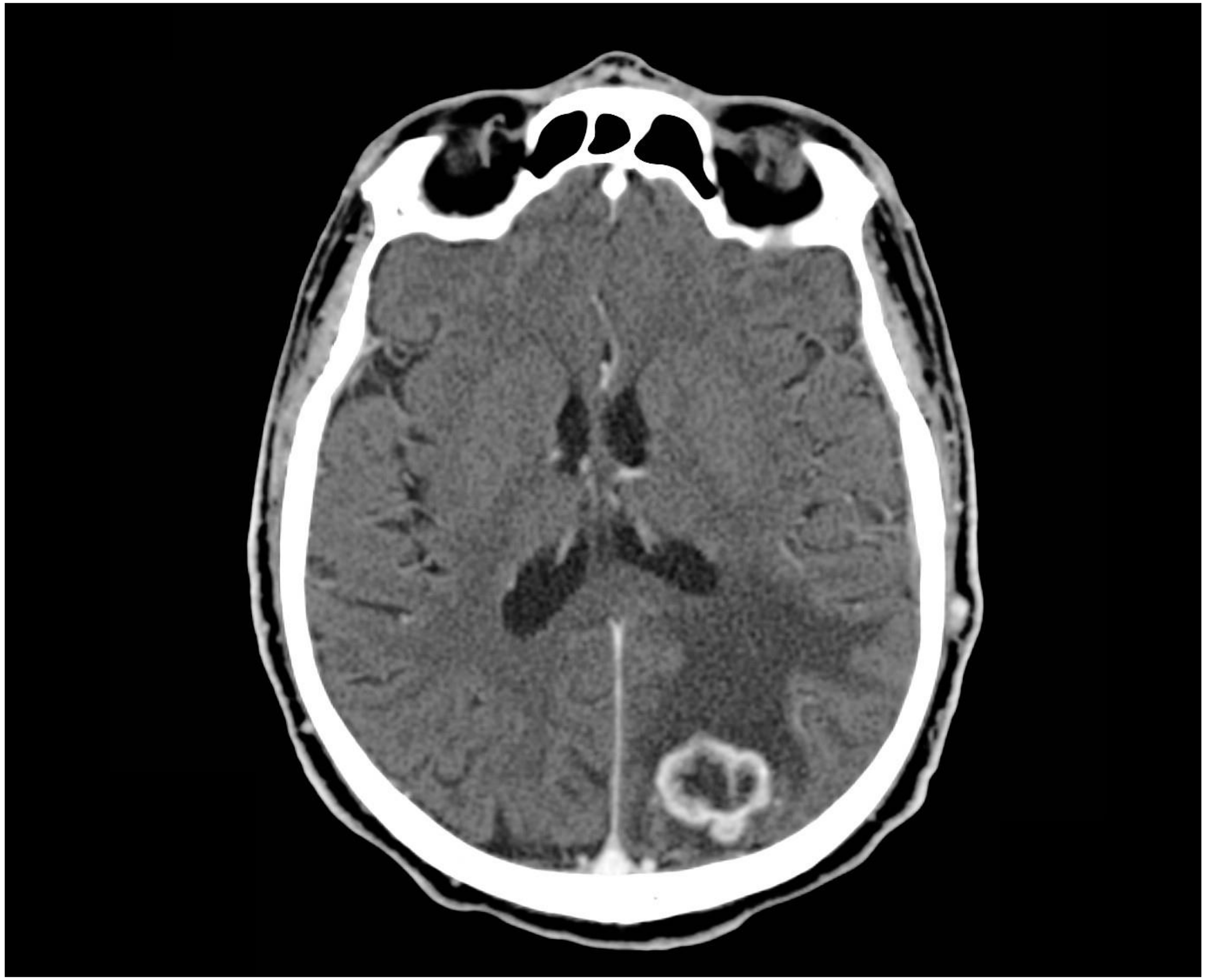


X-ray imaging



# What is Computed Tomography

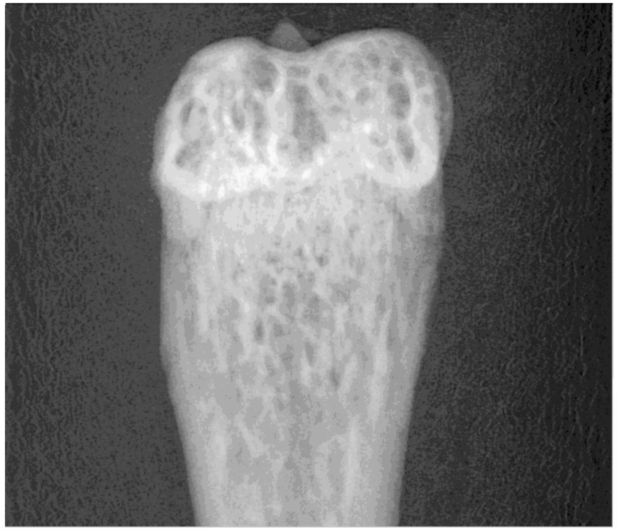




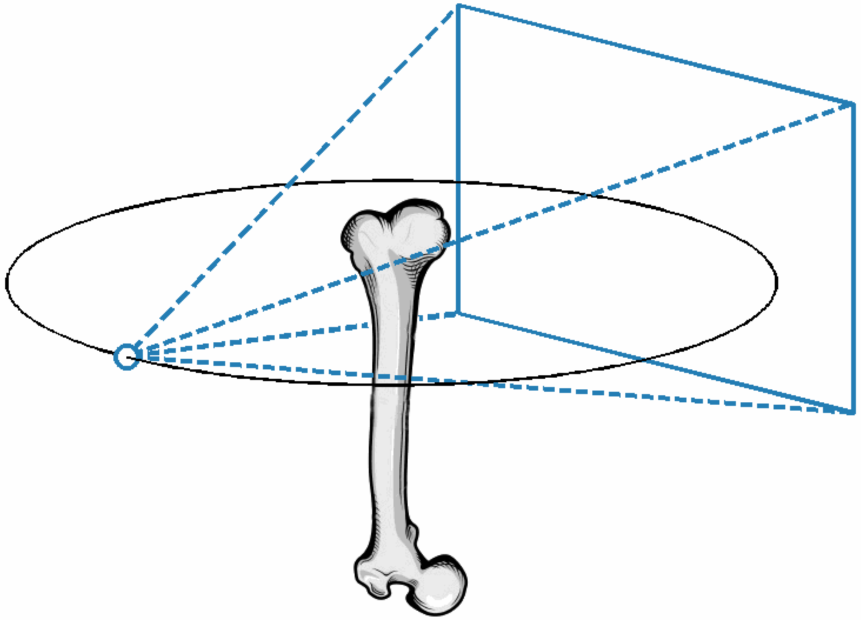
X-ray imaging

# What is Computed Tomography





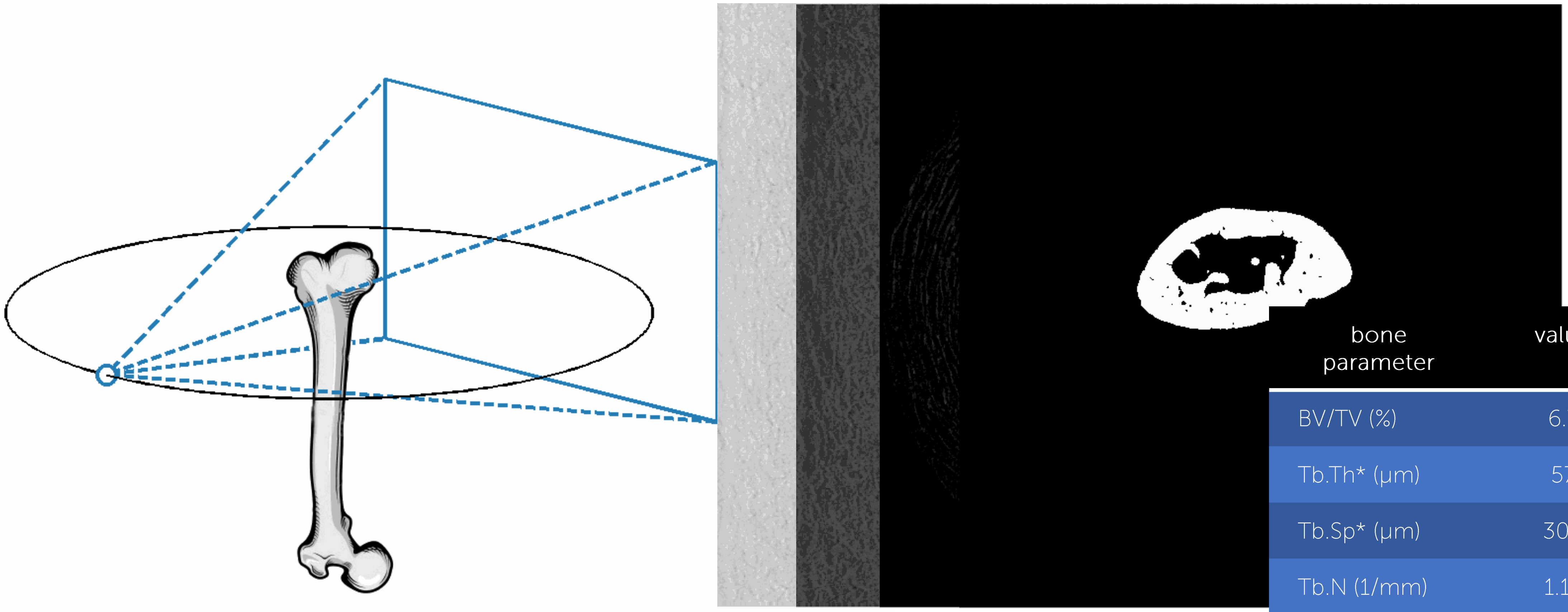
projection



reconstruction

# A typical CT workflow





| bone parameter | value |
|---|---|
| BV/TV (%) | 6.7 |
| Tb.Th* (μm) | 57 |
| Tb.Sp* (μm) | 306 |
| Tb.N (1/mm) | 1.17 |
| SMI | 2.5 |

acquisition → data preprocessing → reconstruction → segmentation → analysis

# Overview

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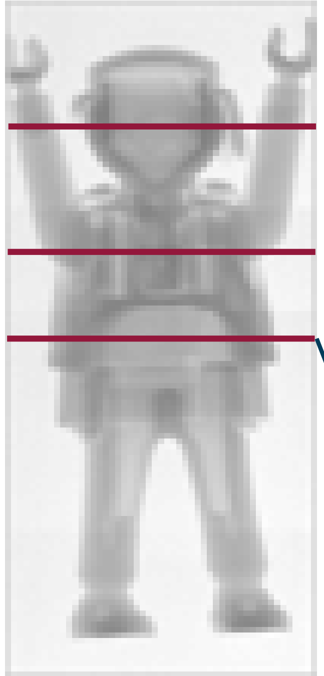
# Medical CT

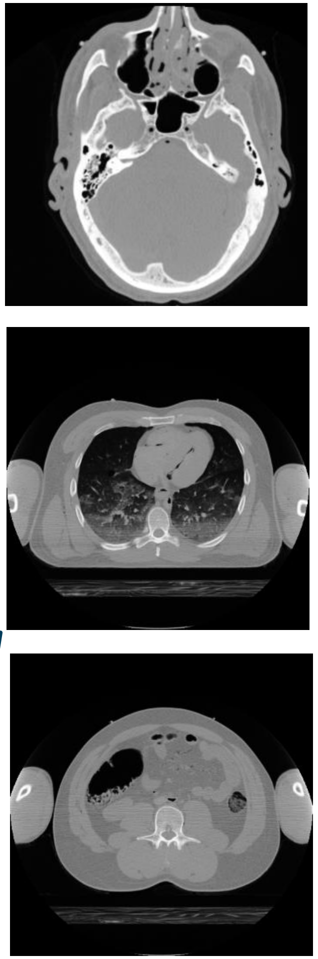


patient


medical CT scanner



projections

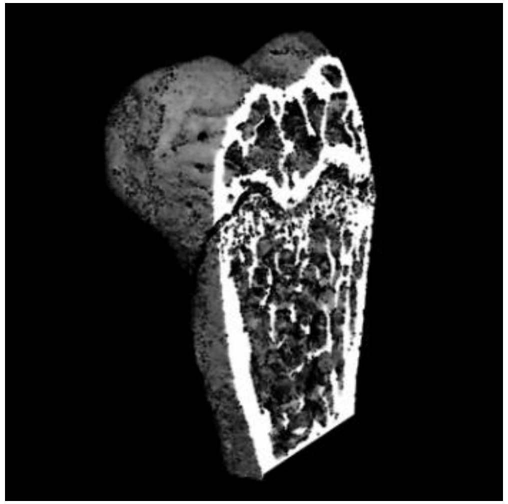


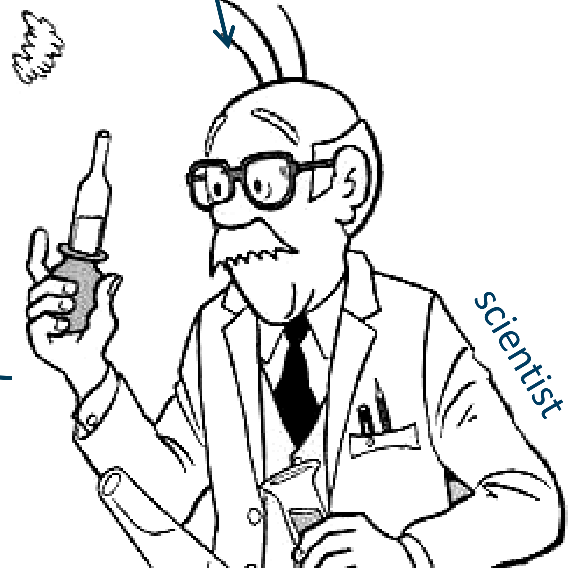
# Biomedical CT

small animal

micro-CT scanner

reconstruction of femur

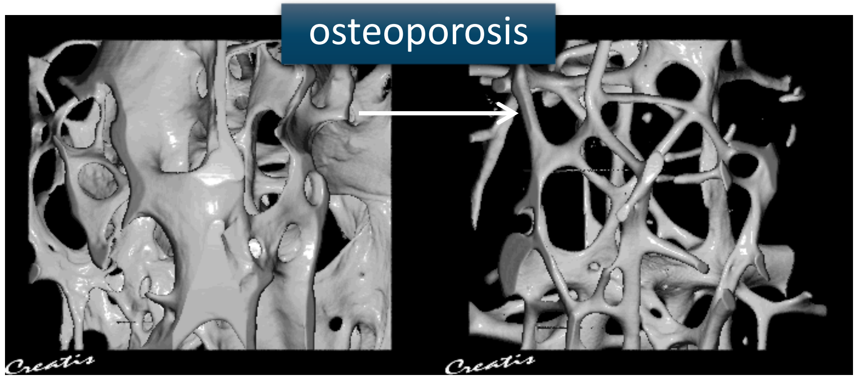




scientist

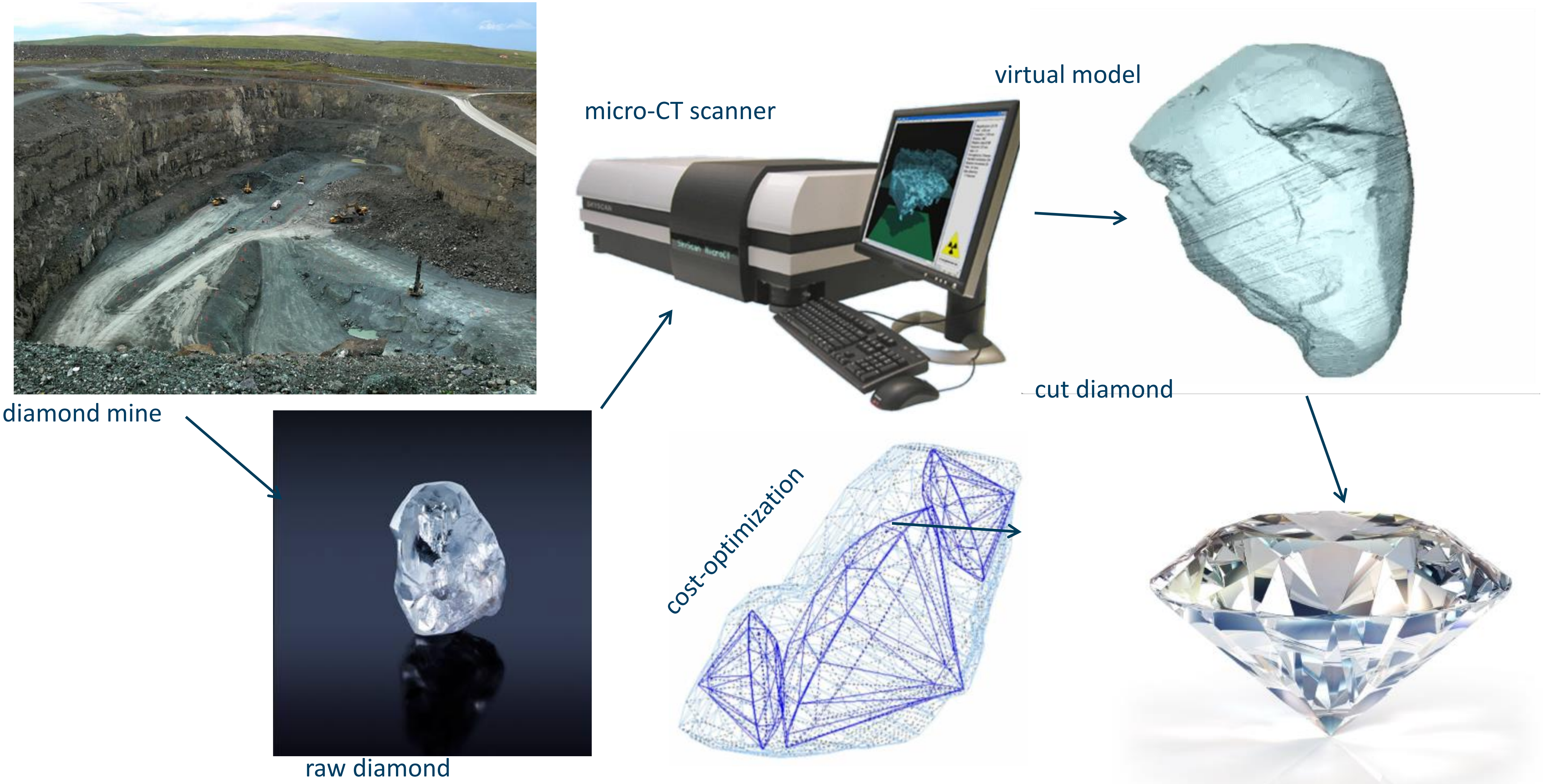
medicine

osteoporosis

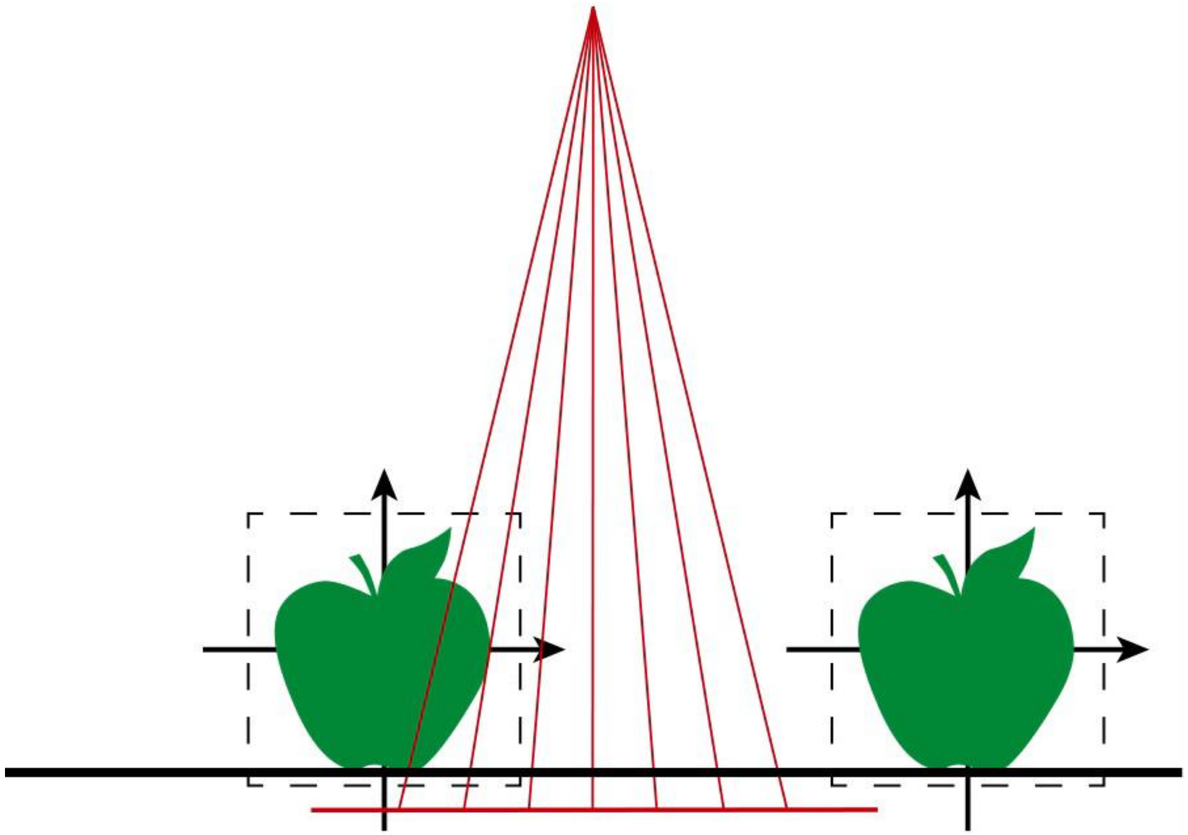


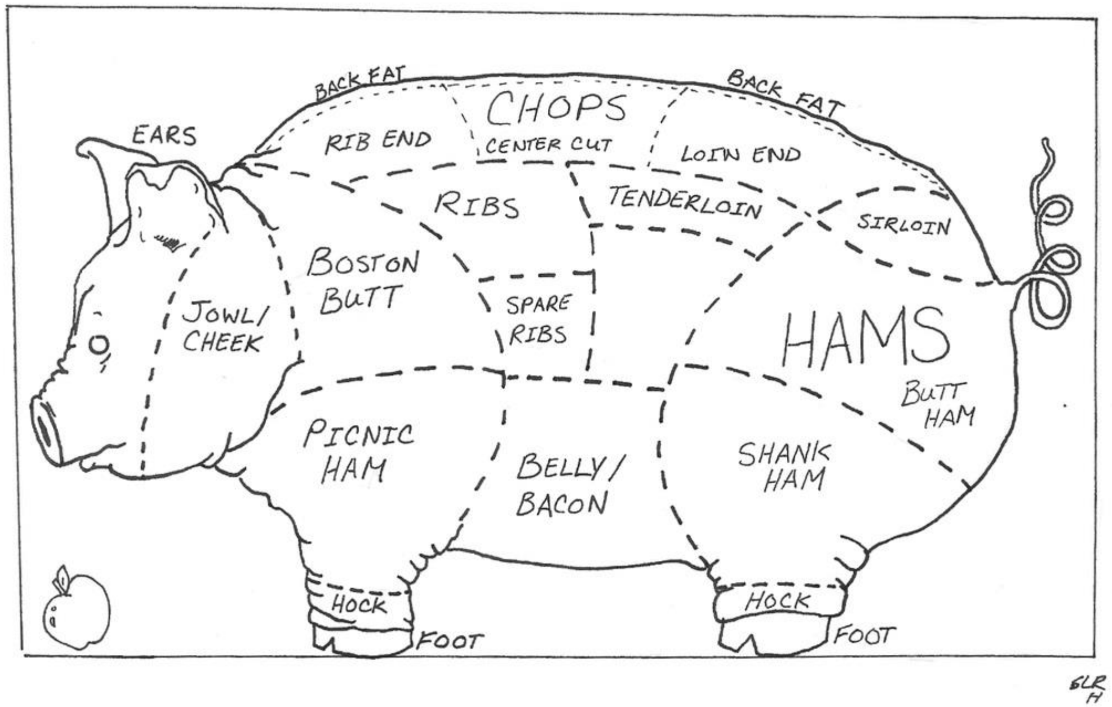
# Industrial CT





# Food Processing

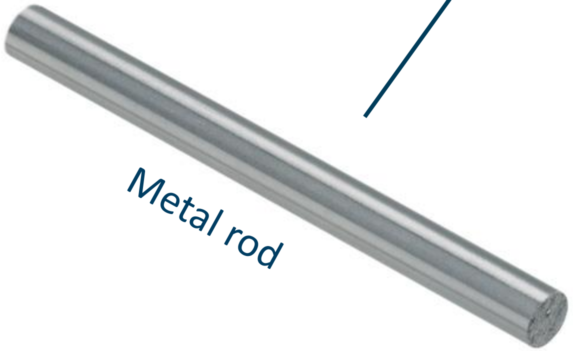




# Material Science

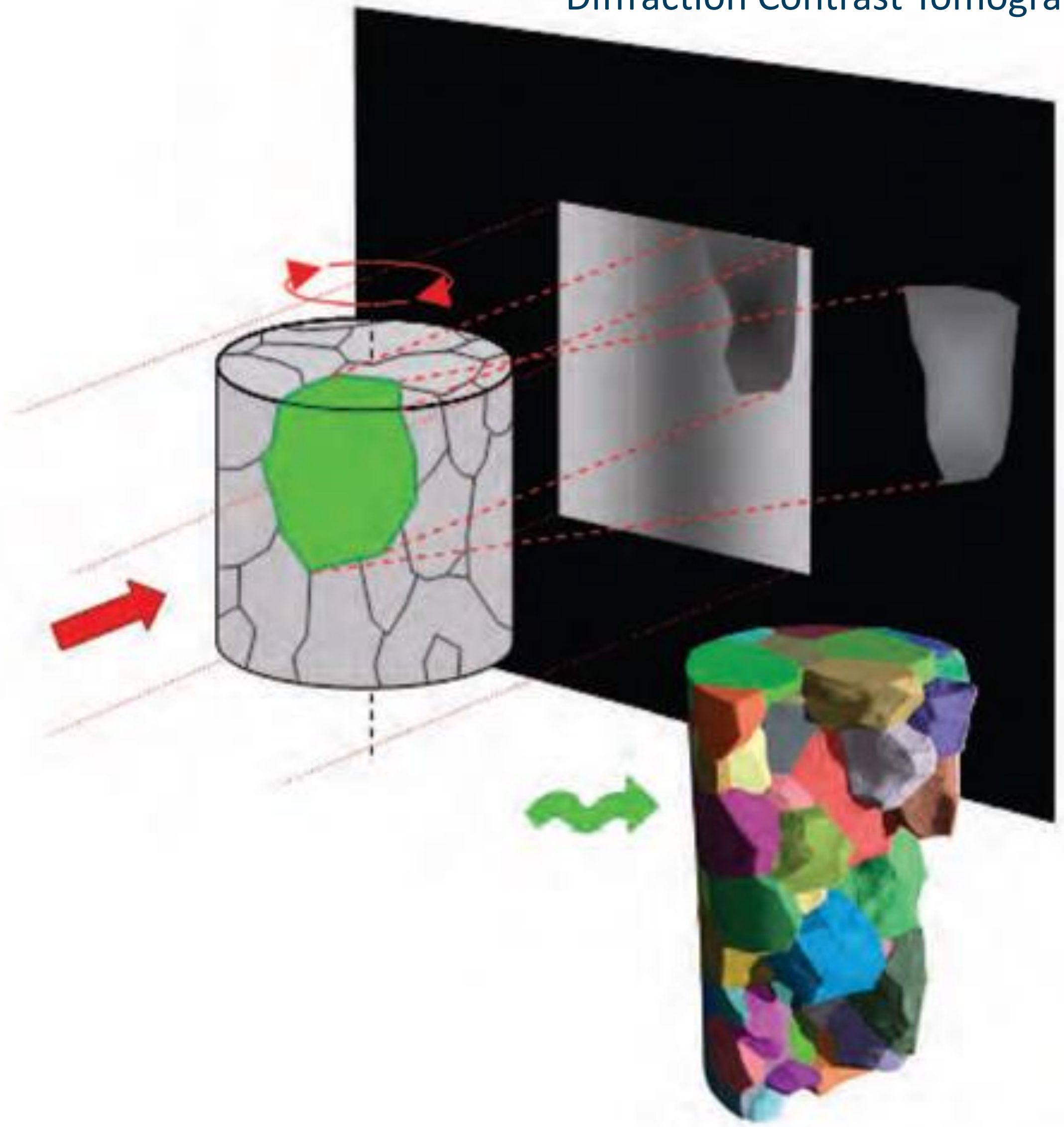


Synchrotron facility



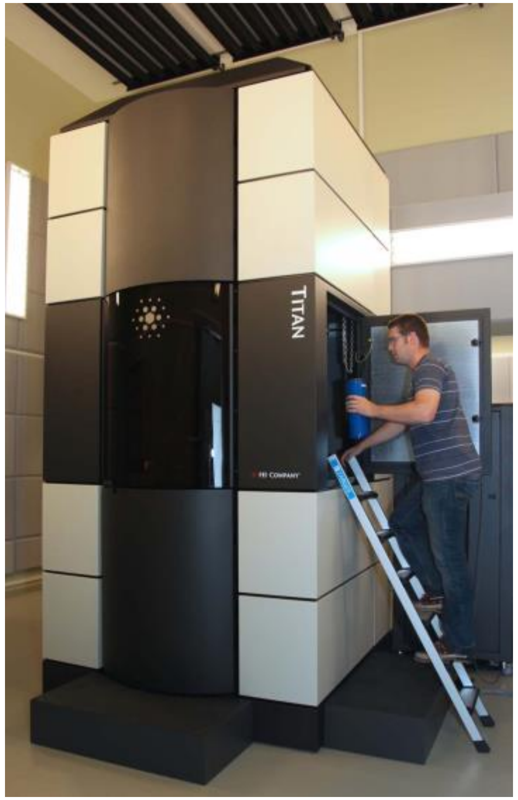
Metal rod

Diffraction Contrast Tomography



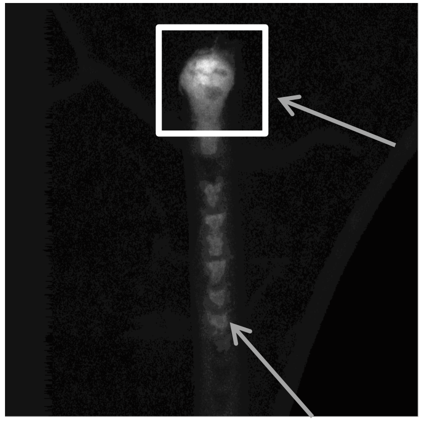
# Nano CT






electron microscope

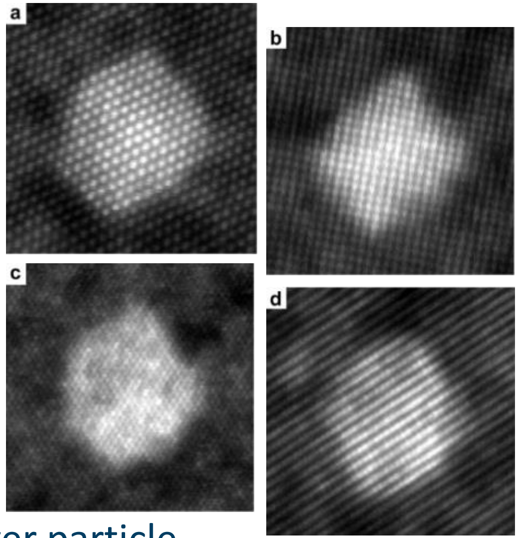
projections



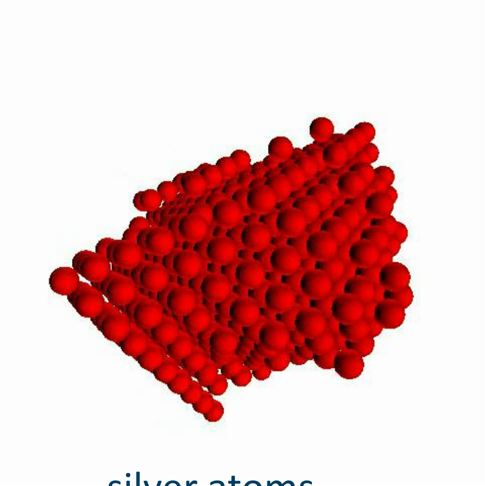
catalyst particle

bamboo-like carbon nanotube

chemical composition




silver particle



silver atoms

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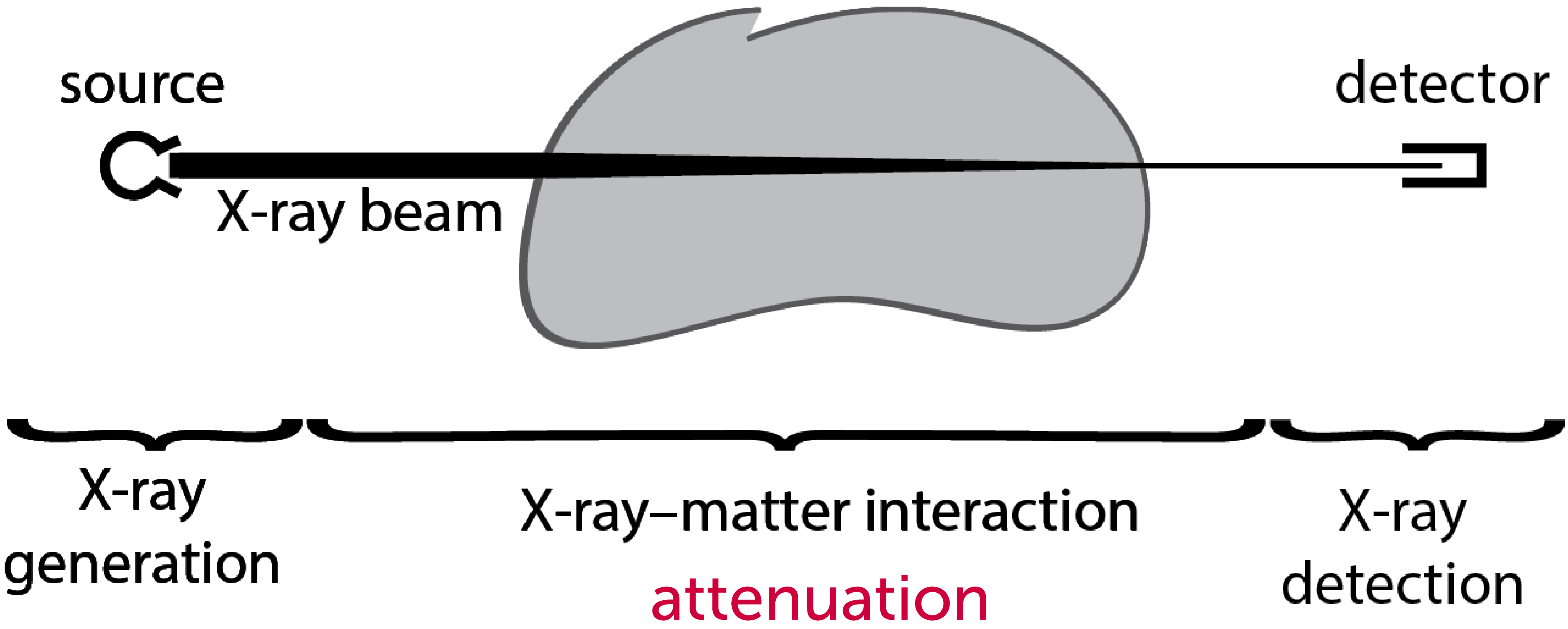
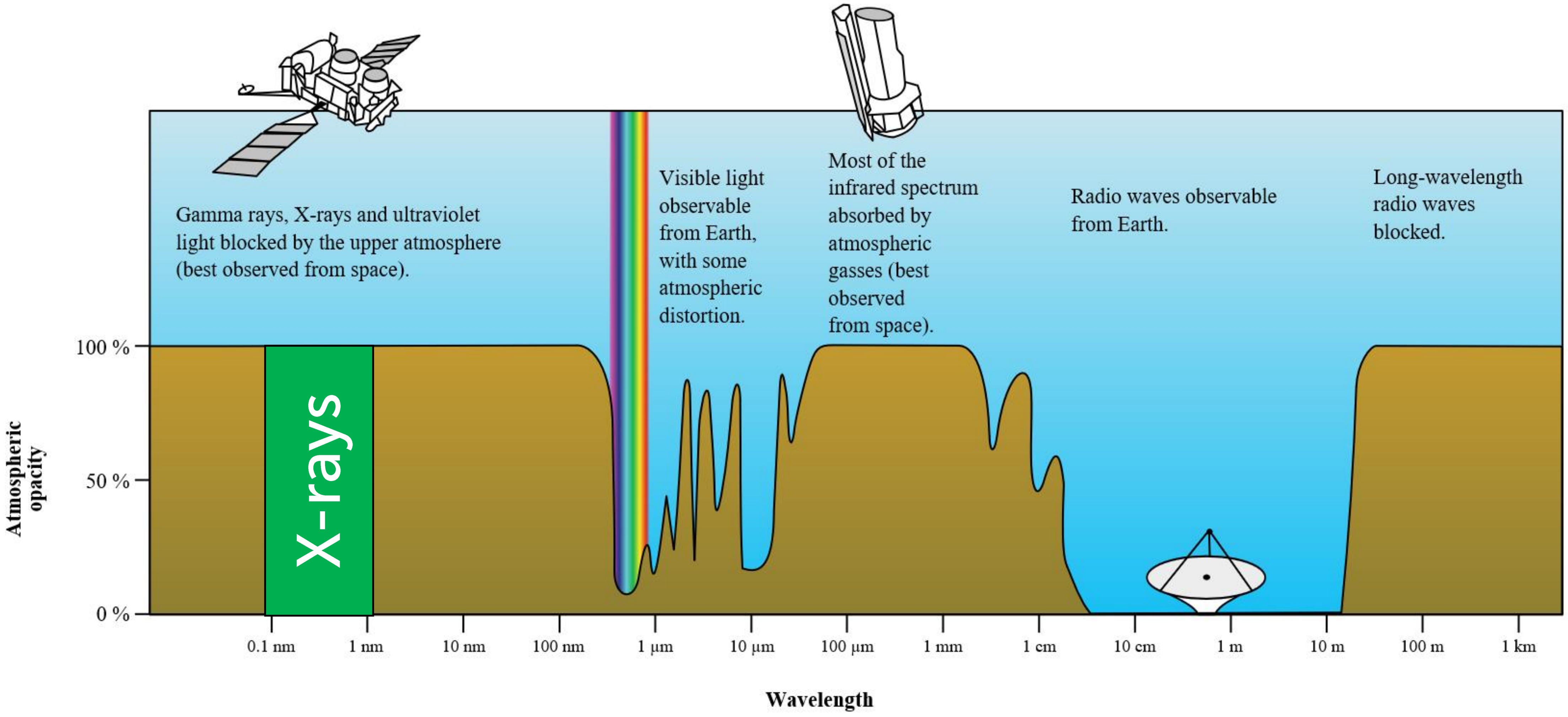
# Basic X-ray physics

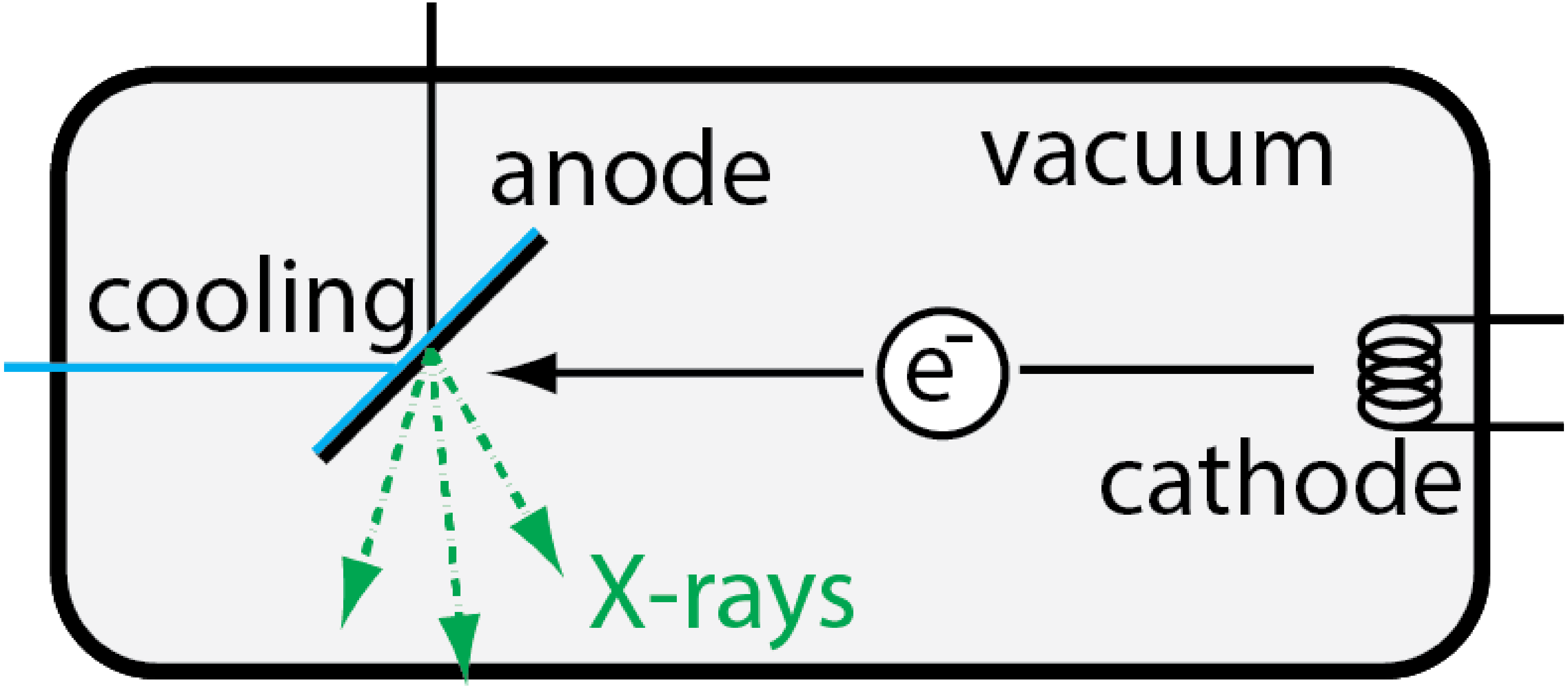




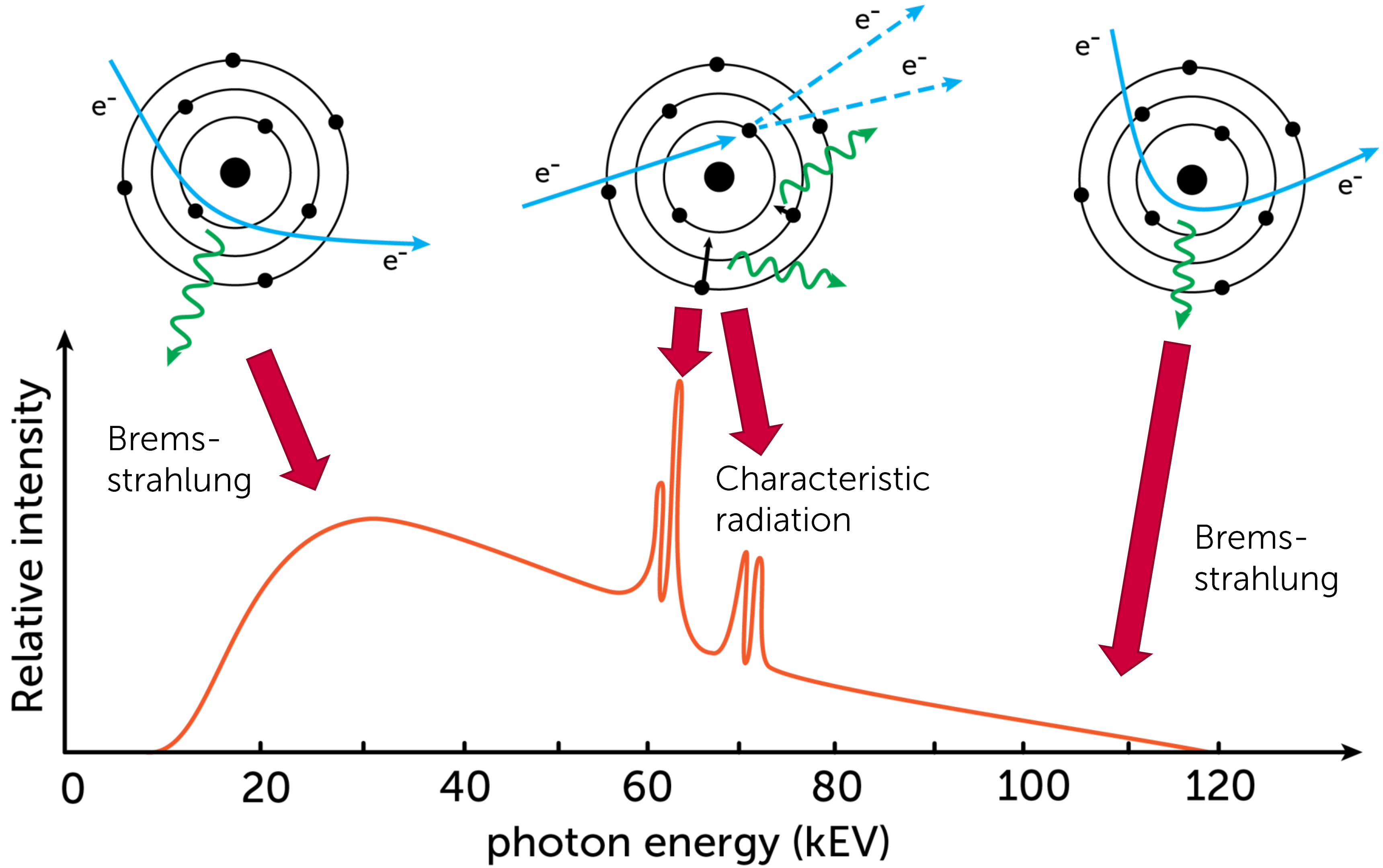
# X-ray generation



## X-ray generation



## X-ray generation - spectrum



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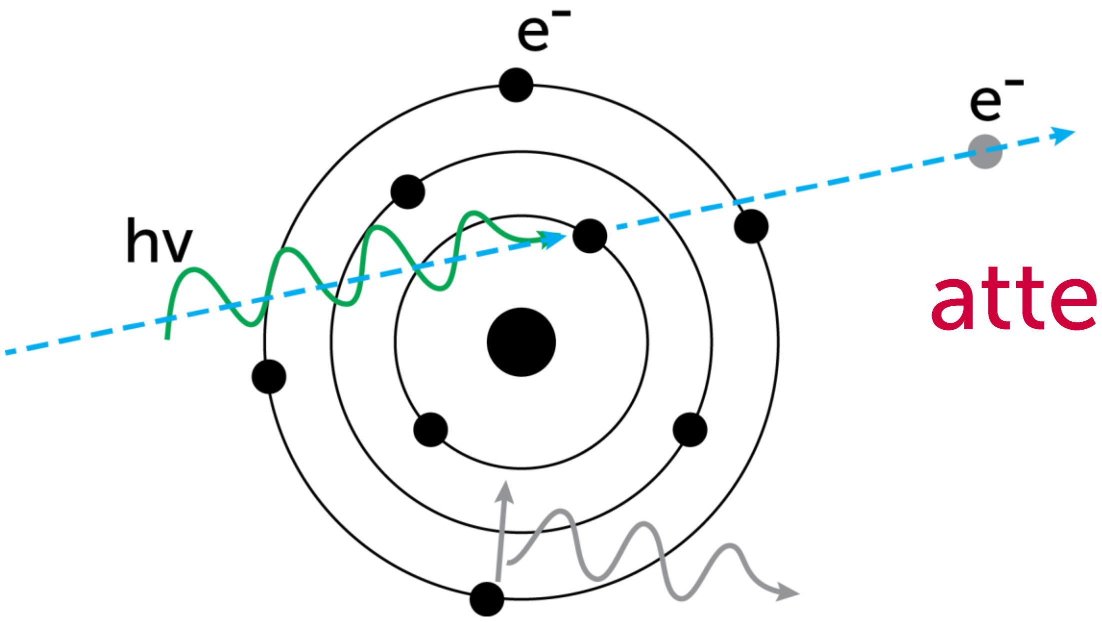
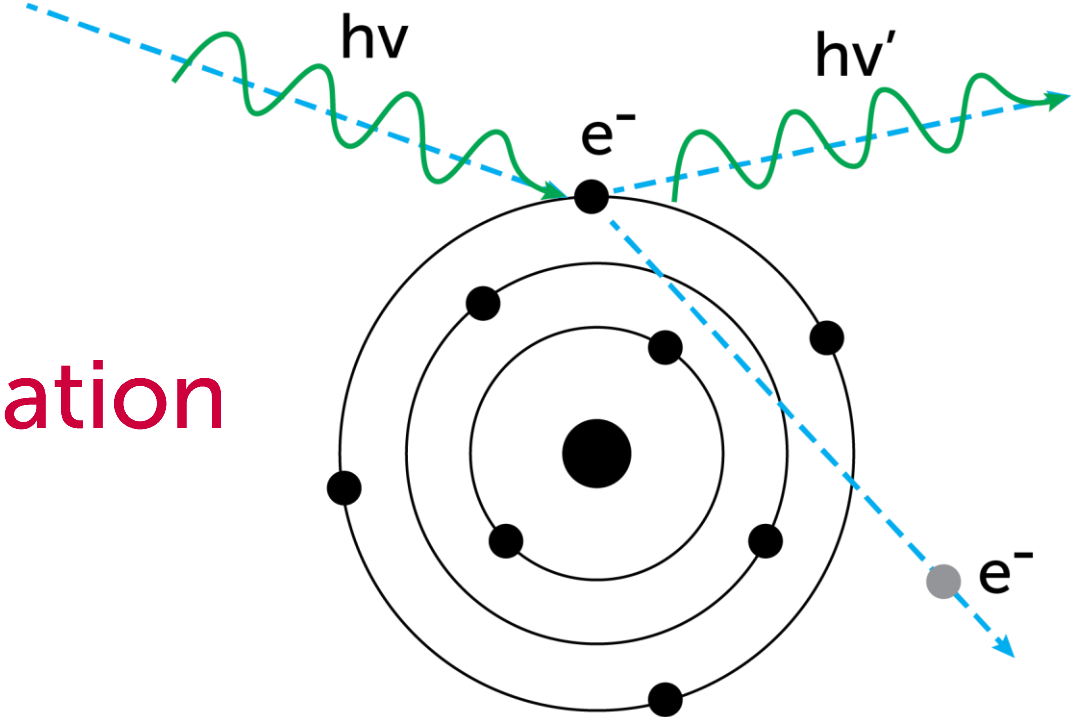
## X-ray-matter interaction



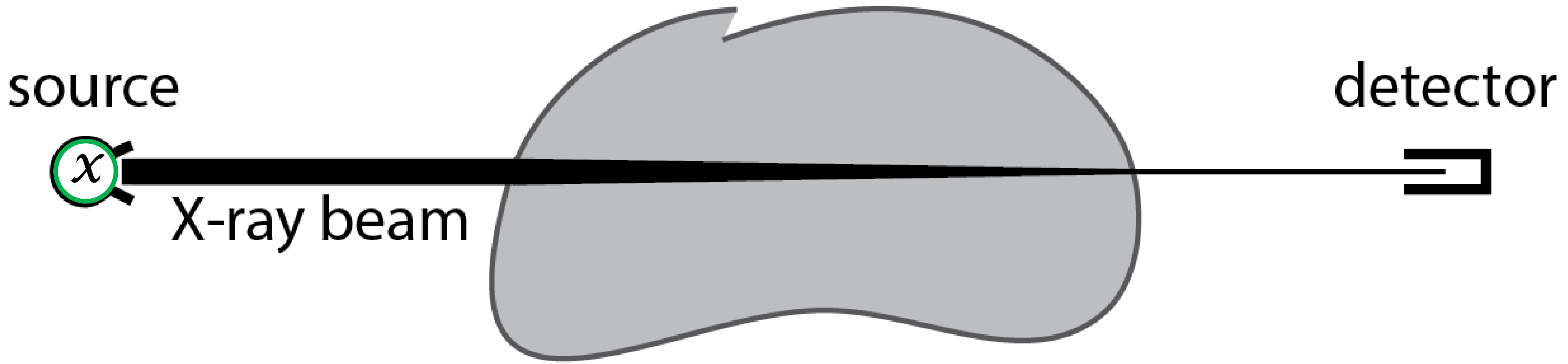

attenuation




photoelectric effect

Compton scattering

## X-ray detection




Beam intensity at source

$$I_0(E)$$

Beam intensity at detector

$$I(E) = I_0(E)e^{-\int \mu(\xi,E)d\xi}$$

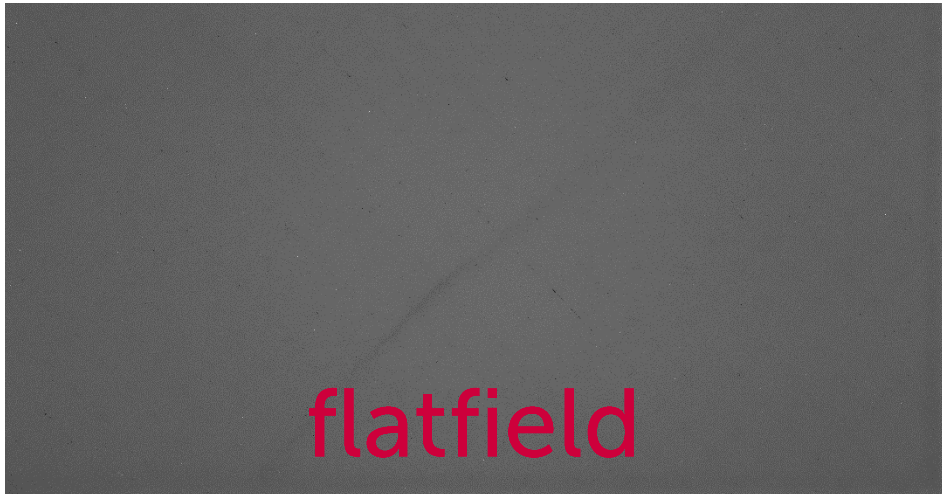
Beam intensity changes

$$I(\xi + \Delta\xi, E) = I(\xi) - \mu(\xi, E)I(\xi, E)\Delta\xi$$

# Basic X-ray physics




$$I = I_0 e^{-\int \mu(\xi) d\xi}$$




$$I_0$$


$$D_0$$


Law of Beer-Lambert: $A = -ln\left(\frac{I-D}{I_0-D}\right)$

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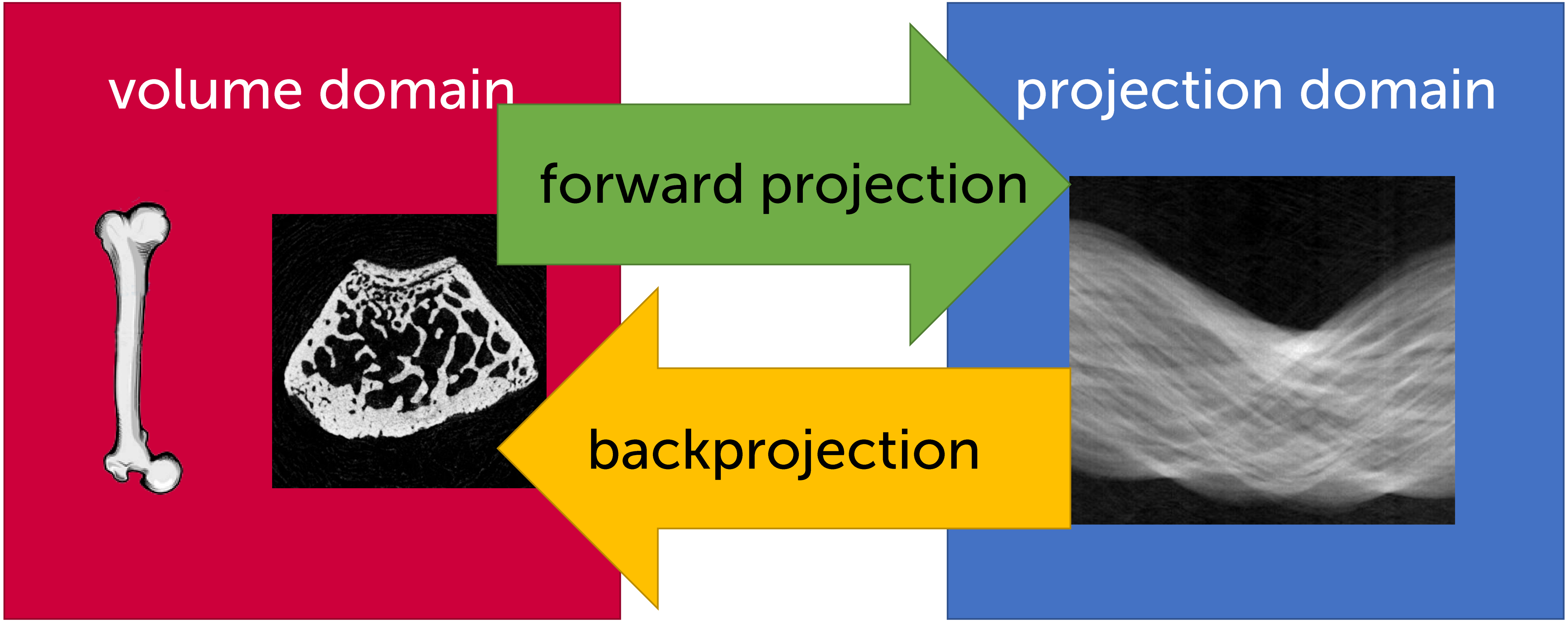
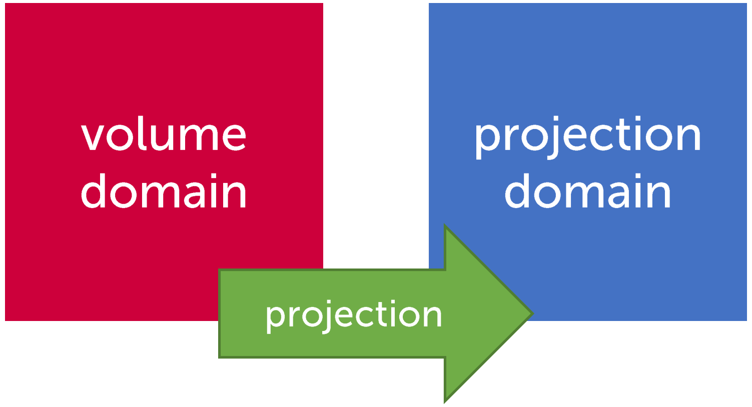
- **A Brief Introduction to Reconstruction Mathematics**
    - **Analytical approach**
    - **Algebraic approach**
- When Physics and Mathematics colide


volume domain
projection domain
forward projection
backprojection

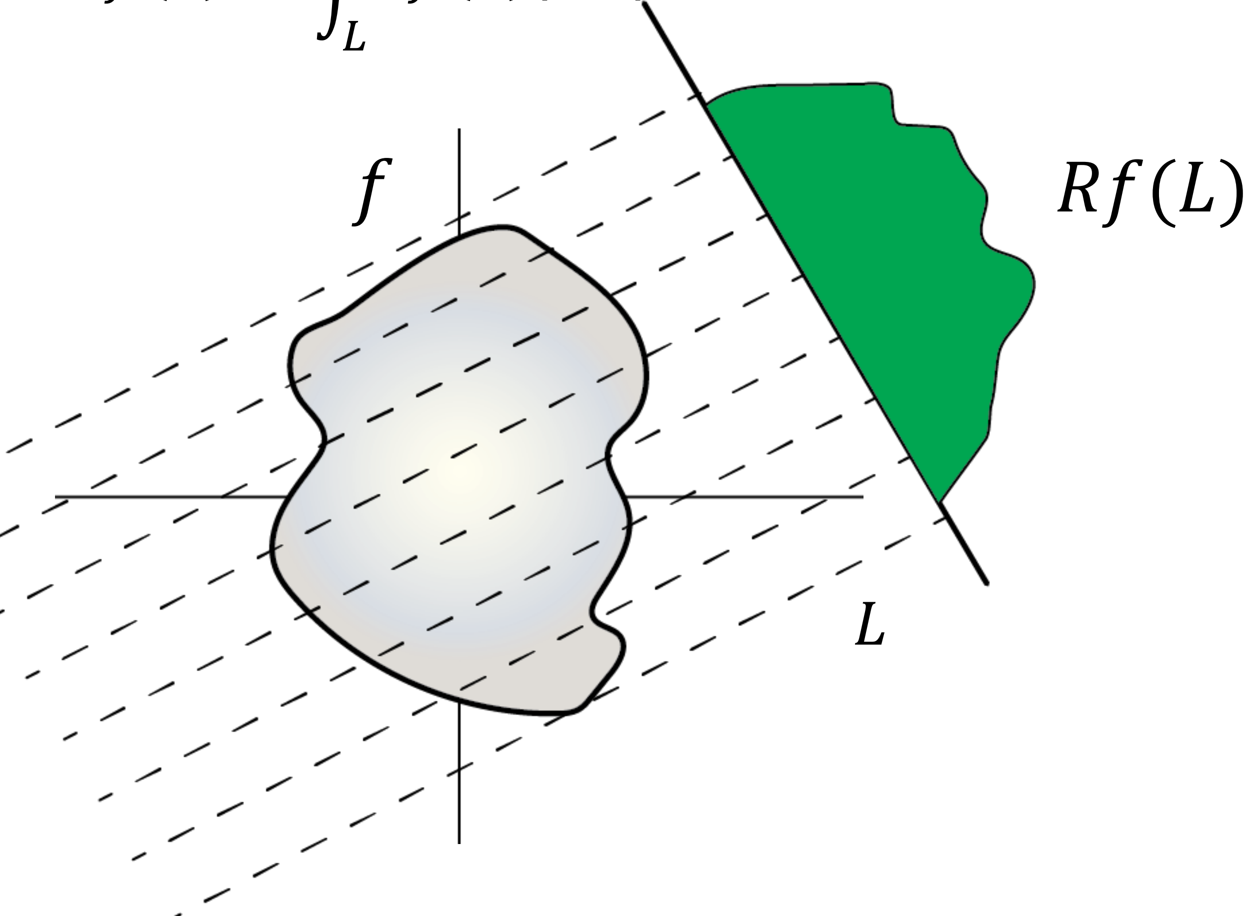
# Basic Mathematics of Computed Tomography





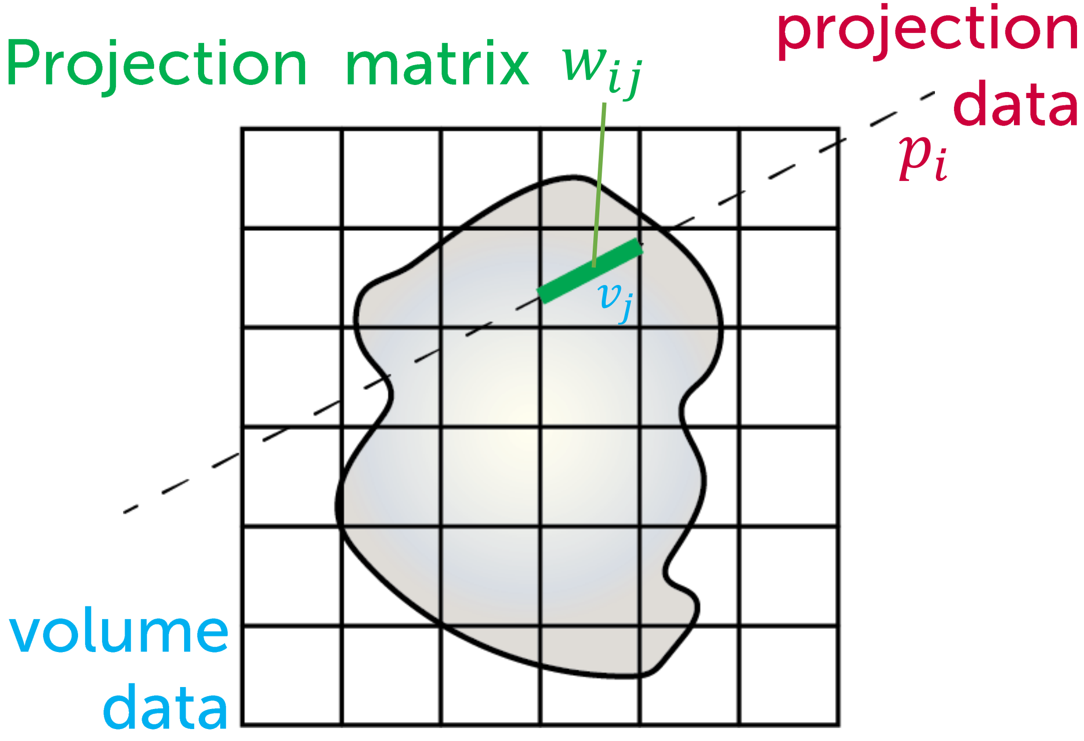

## analytically

$$Rf(L) = \int_L f(x)|dx|$$



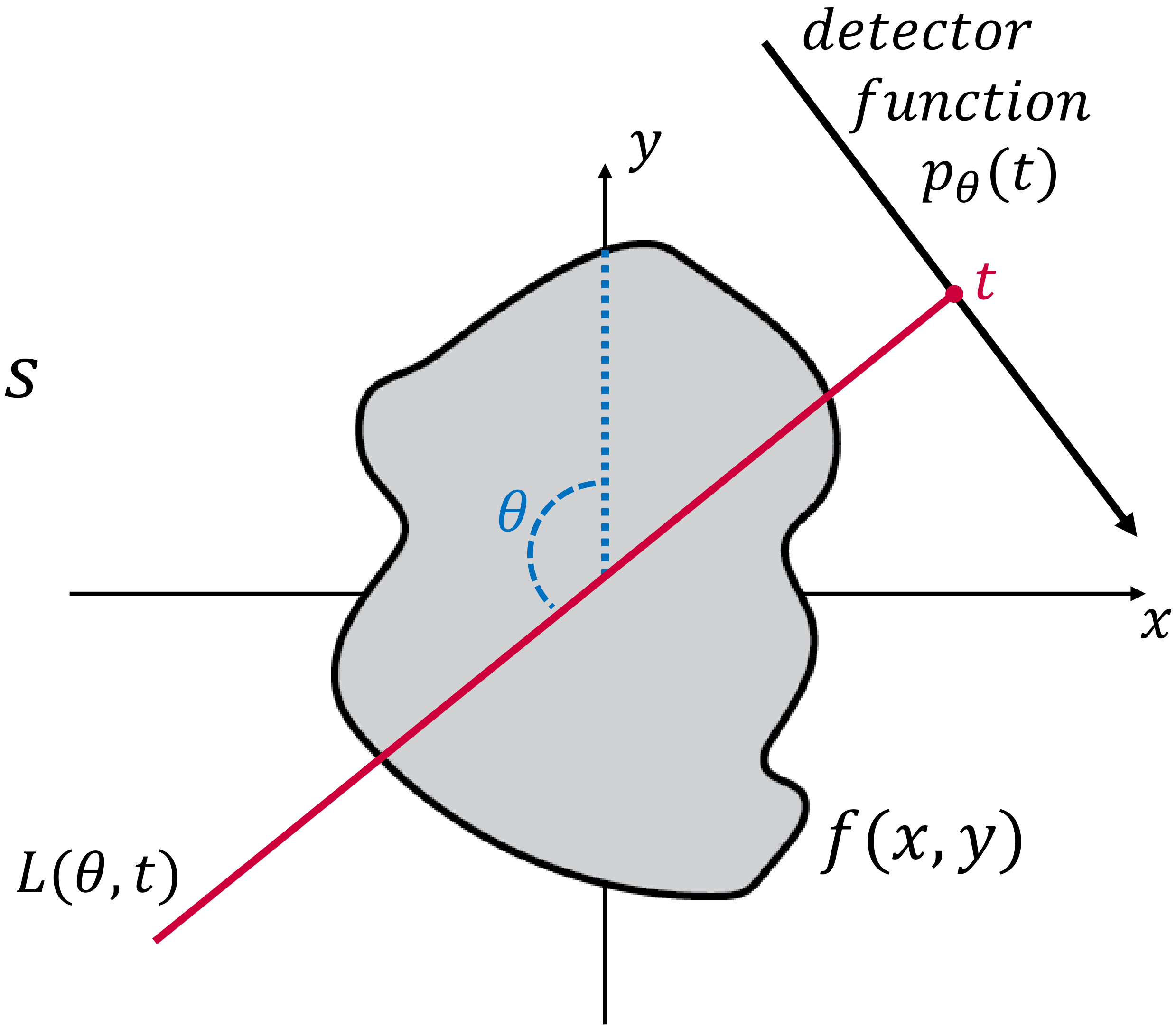

## algebraically

$$p = Wv$$



## Radon transform

$$p_\theta(t) = \int_{L(\theta,t)} f(x,y)\,ds$$

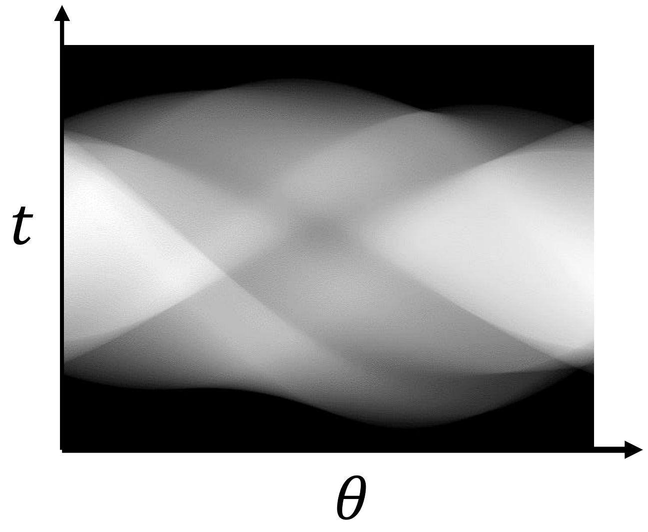


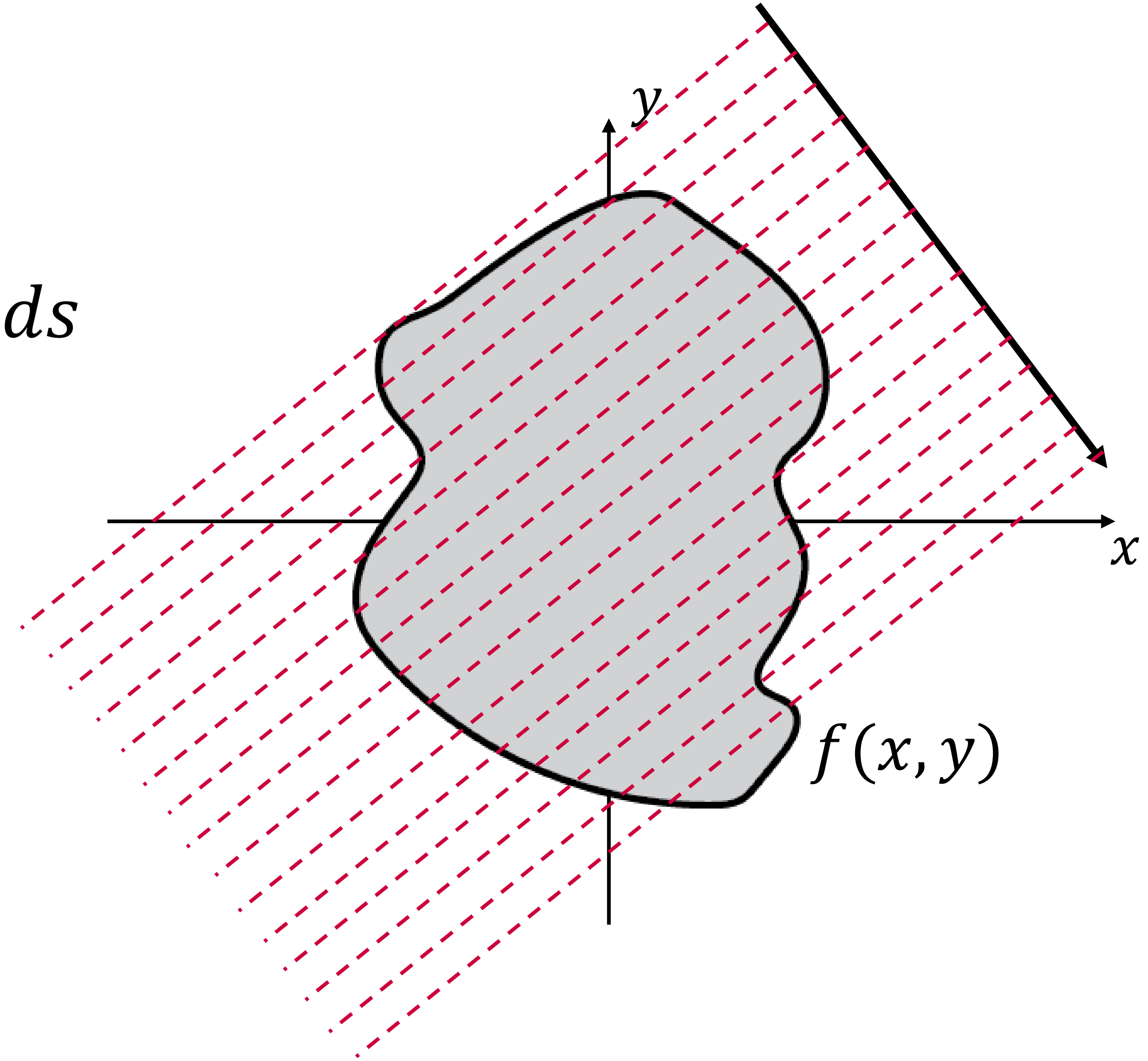

$$L(\theta,t) = \{(x,y) \in \mathbb{R} \times \mathbb{R}: x\cos\theta + y\sin\theta = t\}$$

## Radon transform

$$\mathcal{R}f(\theta,t) = \int_{L(\theta,t)} f(x,y)\,ds$$

sinogram

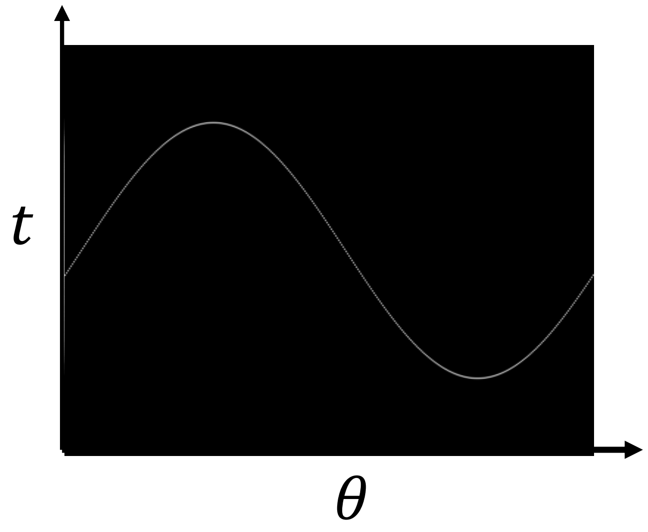


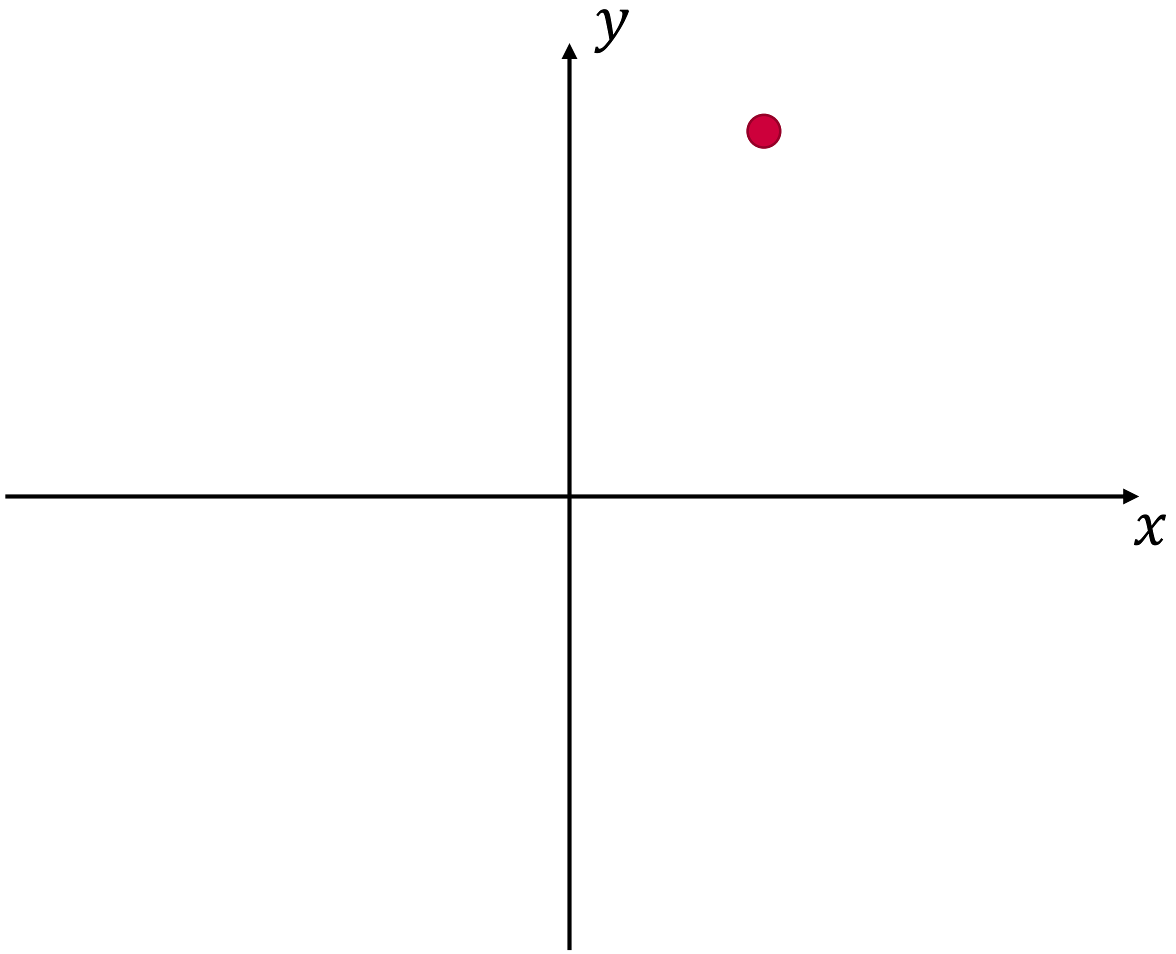



## Radon transform

$$\boxed{\mathcal{R}f(\theta,t)} = \int_{L(\theta,t)} f(x,y)ds$$

sinogram

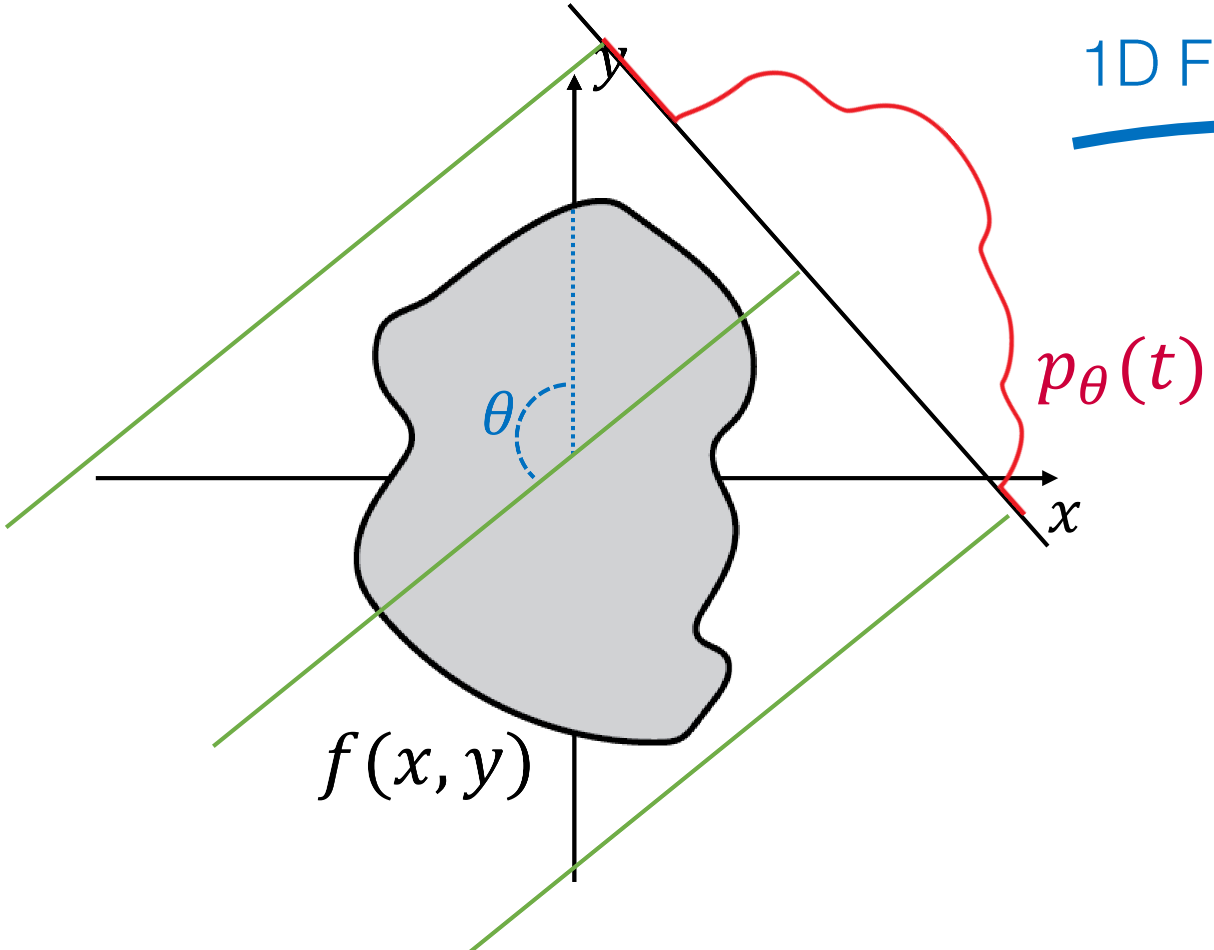


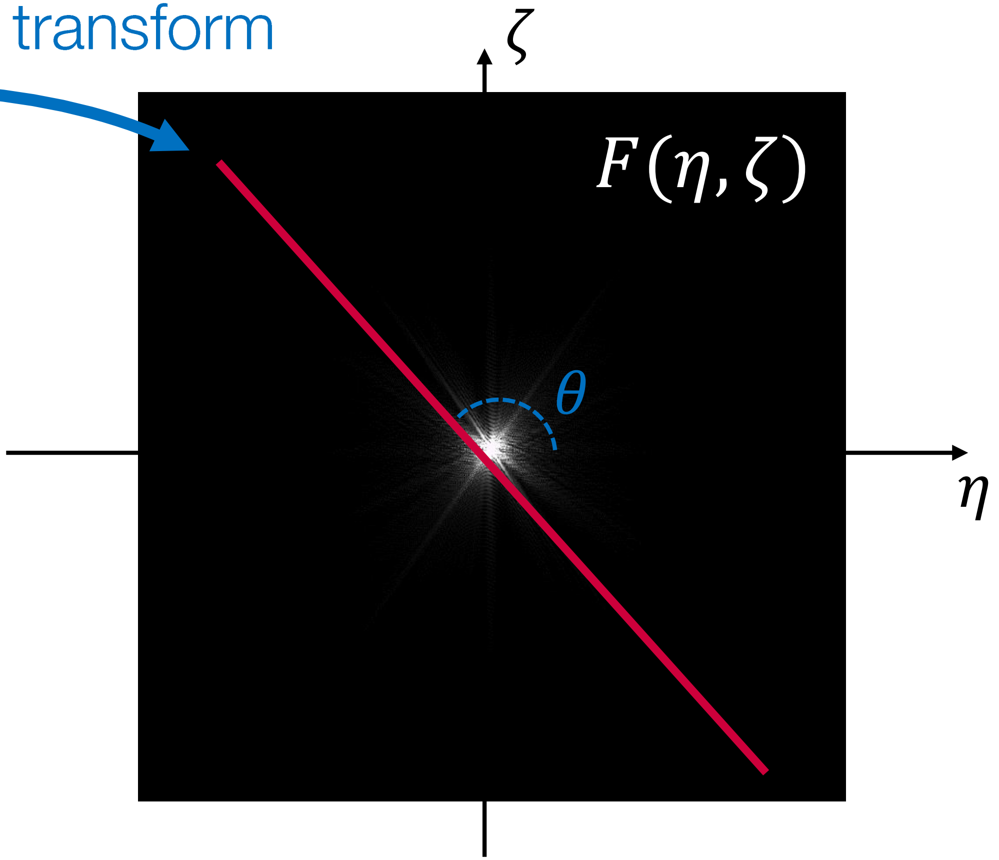
# Fourier Slice Theorem



1D Fourier transform



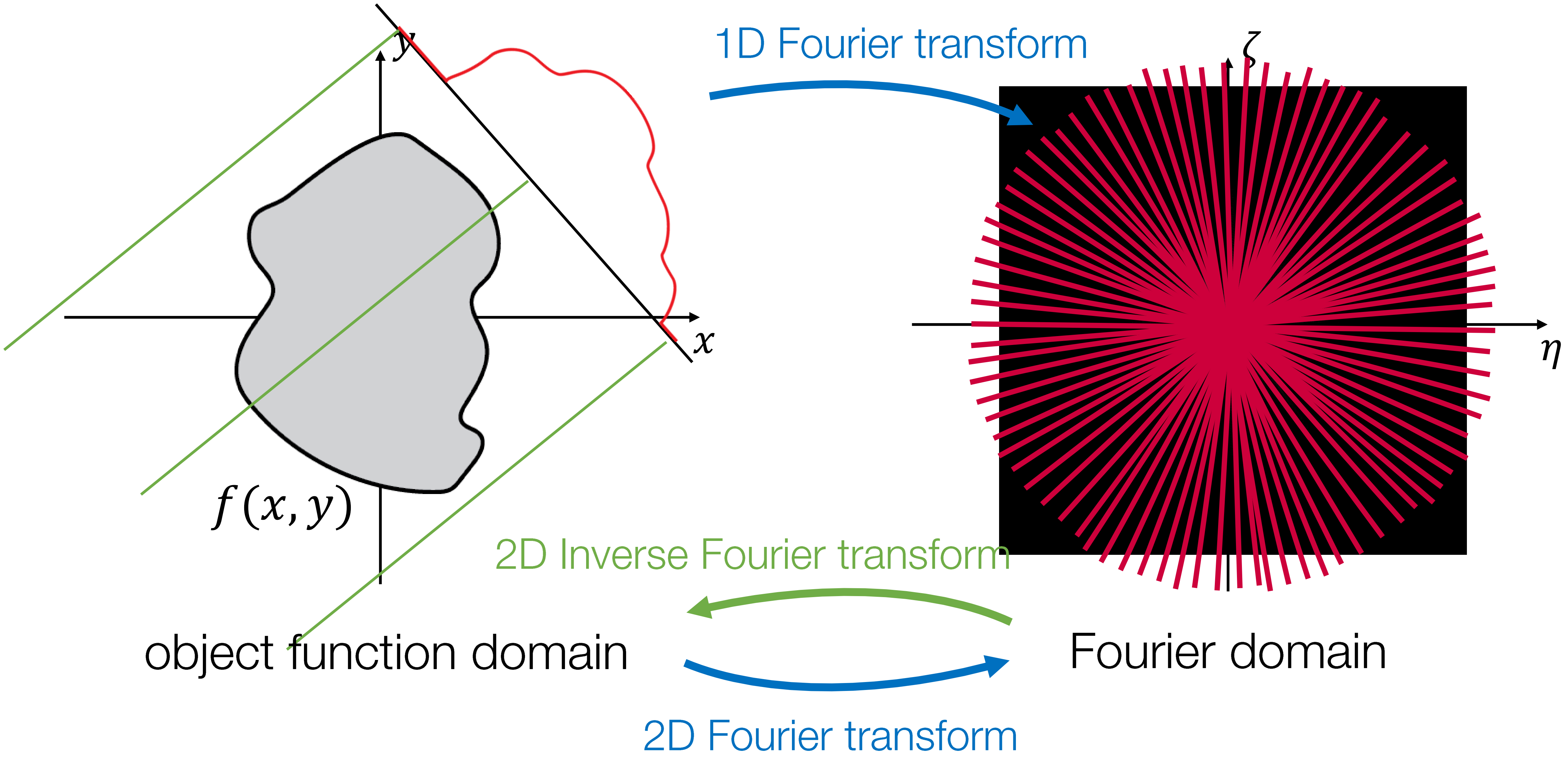

object function domain




Fourier domain

2D Fourier transform

# Fourier Slice Theorem



# Fourier Slice Theorem Proof



The 2D Fourier transform of $f(x, y)$ is:

$$\mathrm{F}(\zeta, \eta) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x, y) e^{-i2\pi(\eta x + \zeta y)} dxdy$$

The 1D Fourier transform of $p_\theta(t)$ is:

$$P_\theta(\omega) = \int_{-\infty}^{\infty} p_\theta(t) e^{-i2\pi\omega t} dt$$

$$P_\theta(\omega) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x, y) e^{-i2\pi\omega(x\cos\theta + y\sin\theta)} dxdy$$

Substituting $\eta = \omega\cos\theta$ and $\zeta = \omega\sin\theta$:

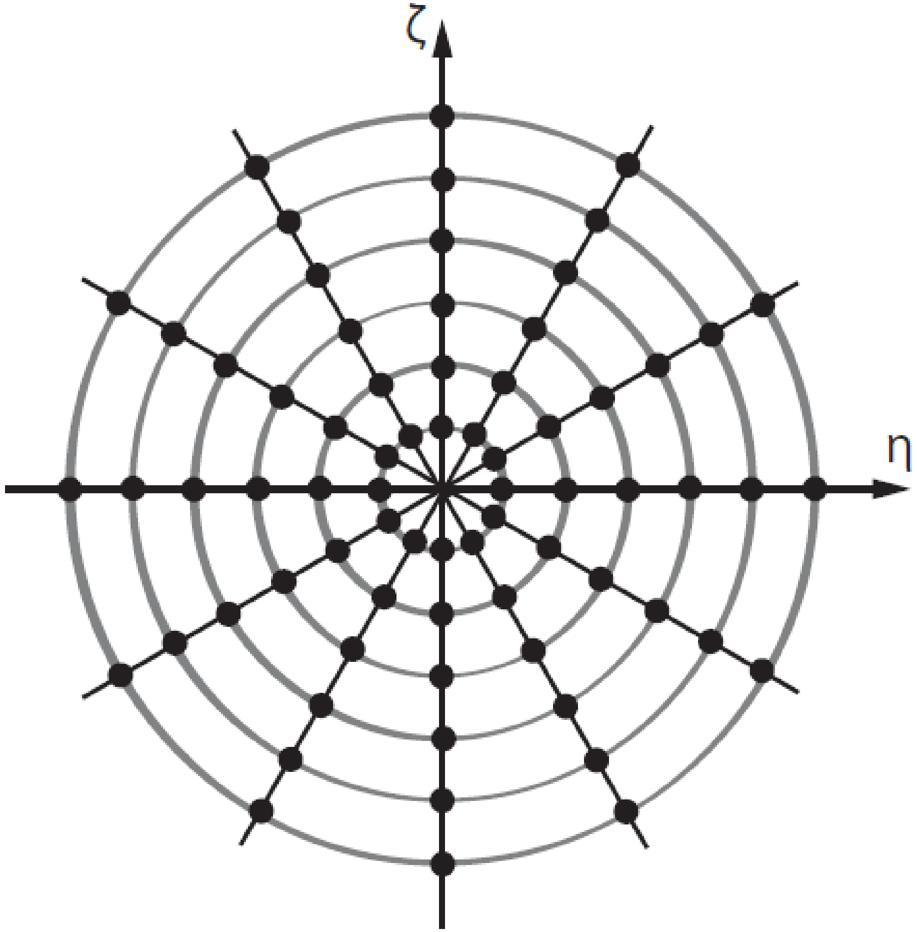
$$p_\theta(t) = \mathcal{R}f(\theta, t) = \int_{L(\theta,t)} f(x, y) ds$$

$$P_\theta(\omega) = \mathrm{F}(\omega\cos\theta, \omega\sin\theta)$$

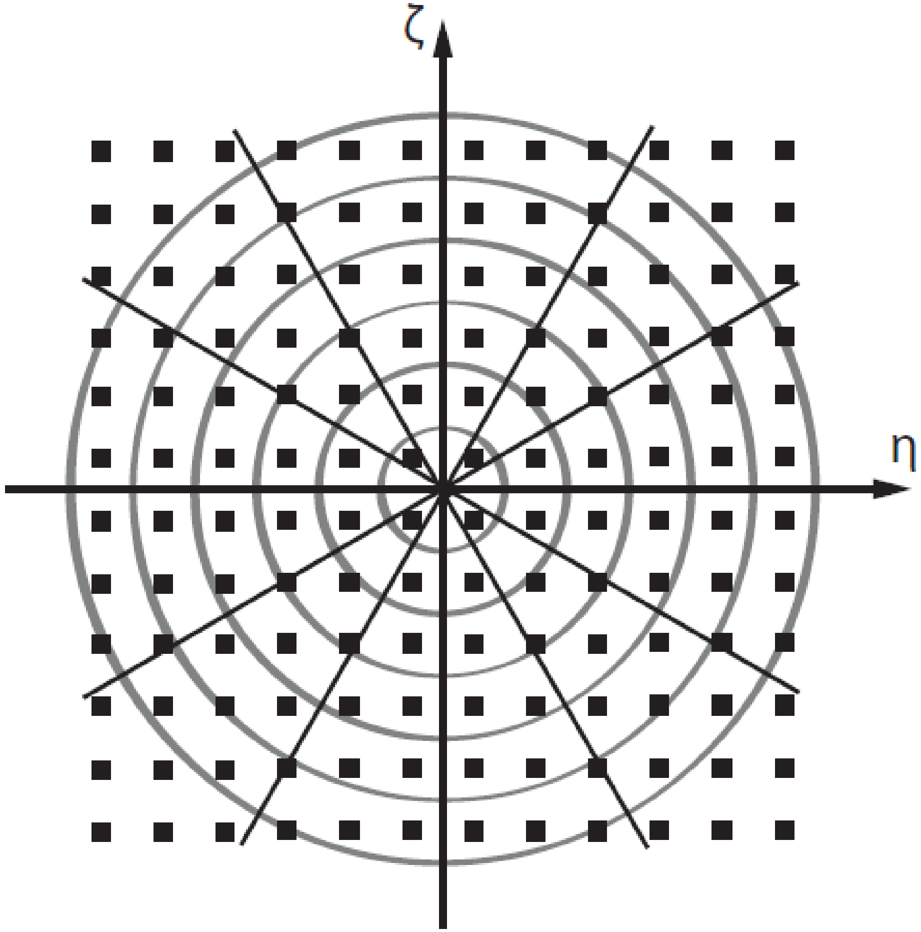
$$L(\theta, t) = \{(x, y) \in \mathbb{R}\times\mathbb{R}: x\cos\theta + y\sin\theta = t\}$$

# Fourier Slice Theorem





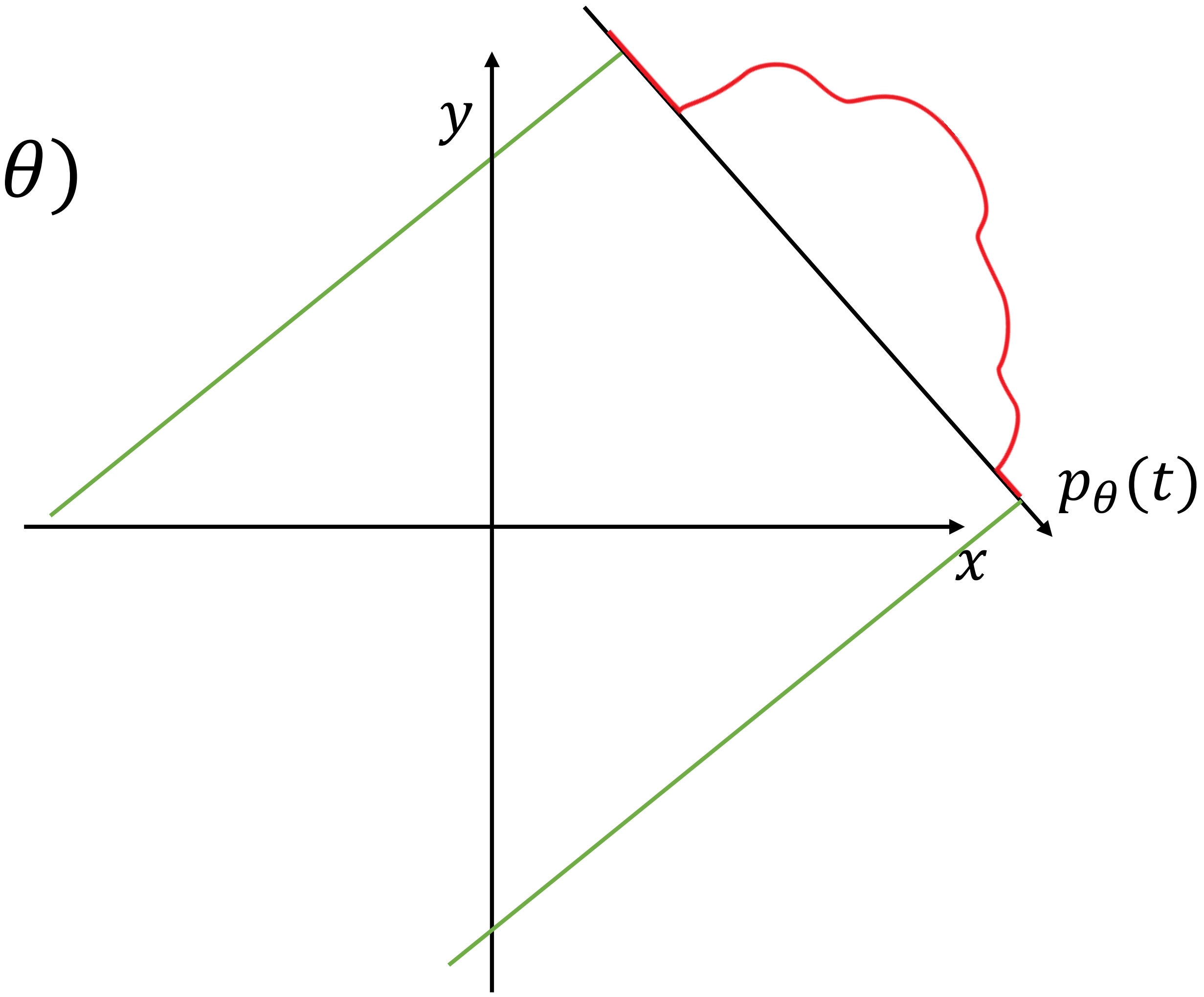

Fourier sampling with Fourier Slice Theorem




Sampling required by Fast Fourier Transform (FFT)

# Filtered Backprojection

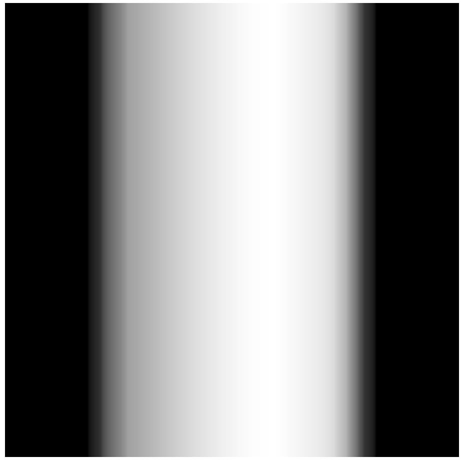
$$f_{bp}(x, y) = \quad p_\theta(x\cos\theta + y\sin\theta)$$



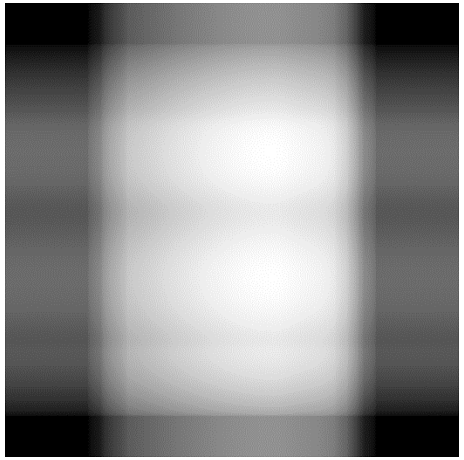
# Filtered Backprojection (FBP)



## backprojection



1 angle



2 angles

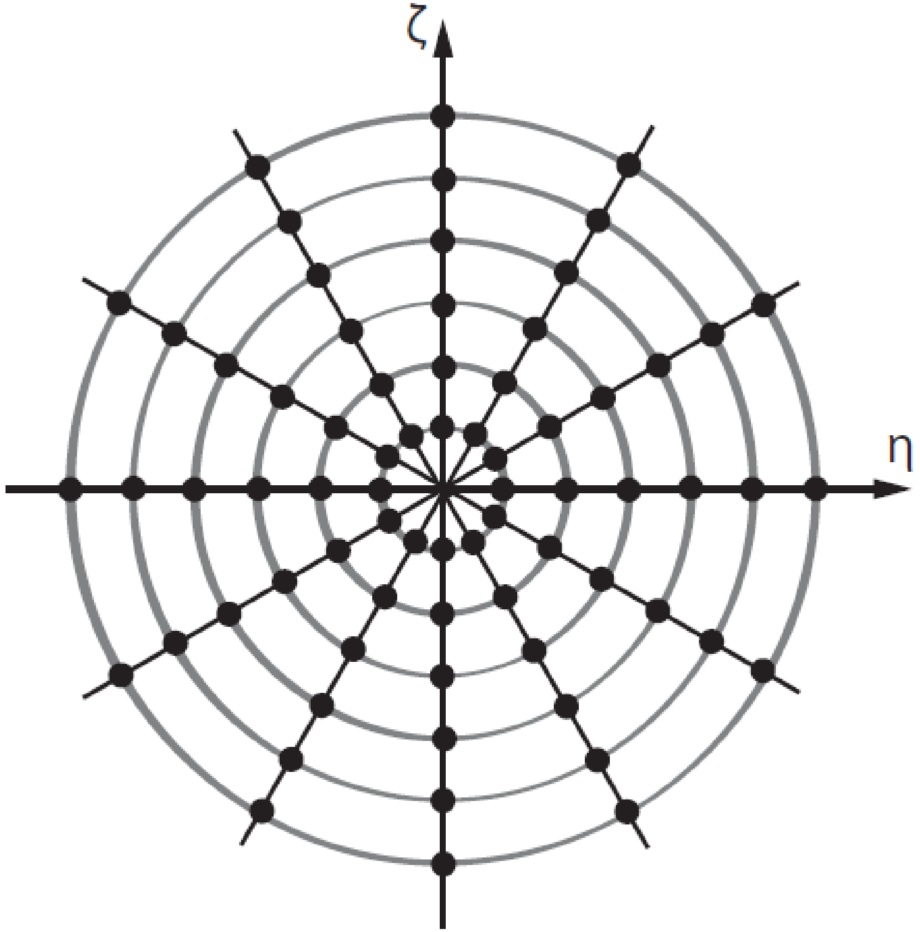
8 angles

45 angles

180 angles

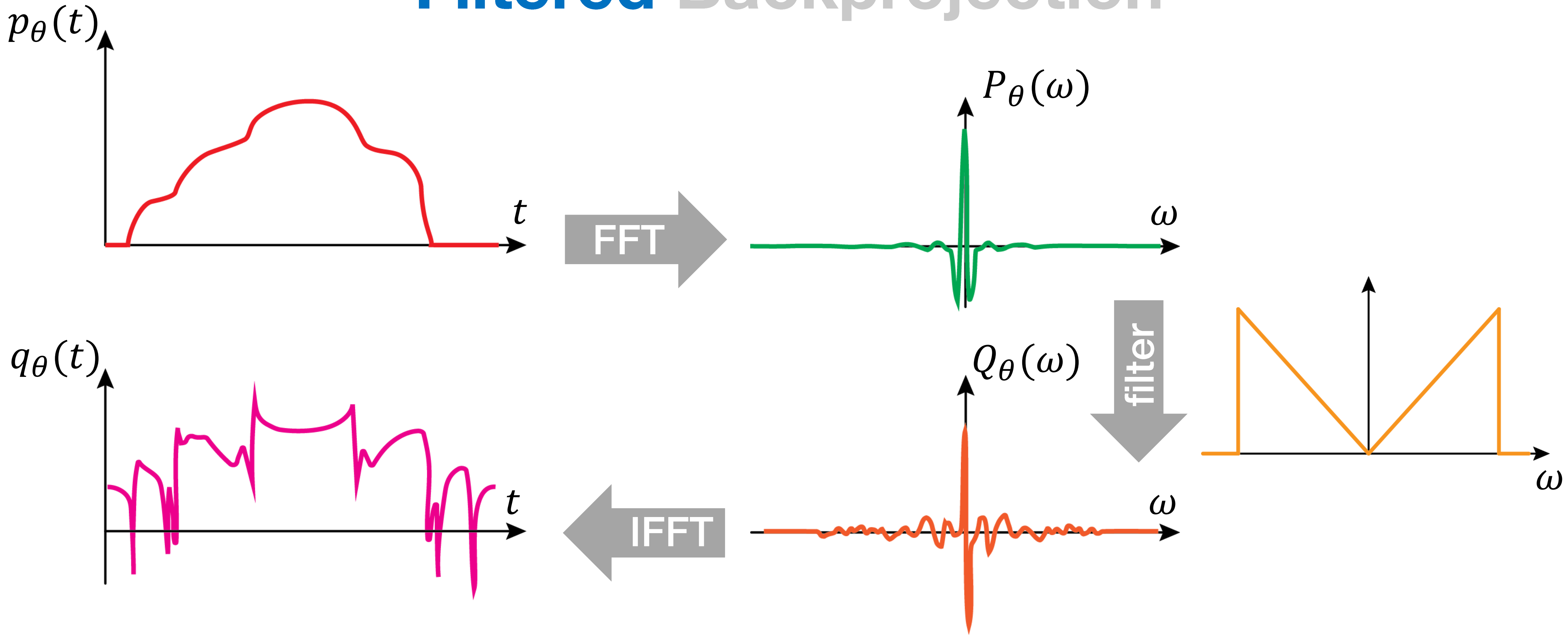
# Filtered Backprojection (FBP)






Sampling of Fourier domain

## **Filtered** Backprojection



# Filtered Backprojection (FBP)

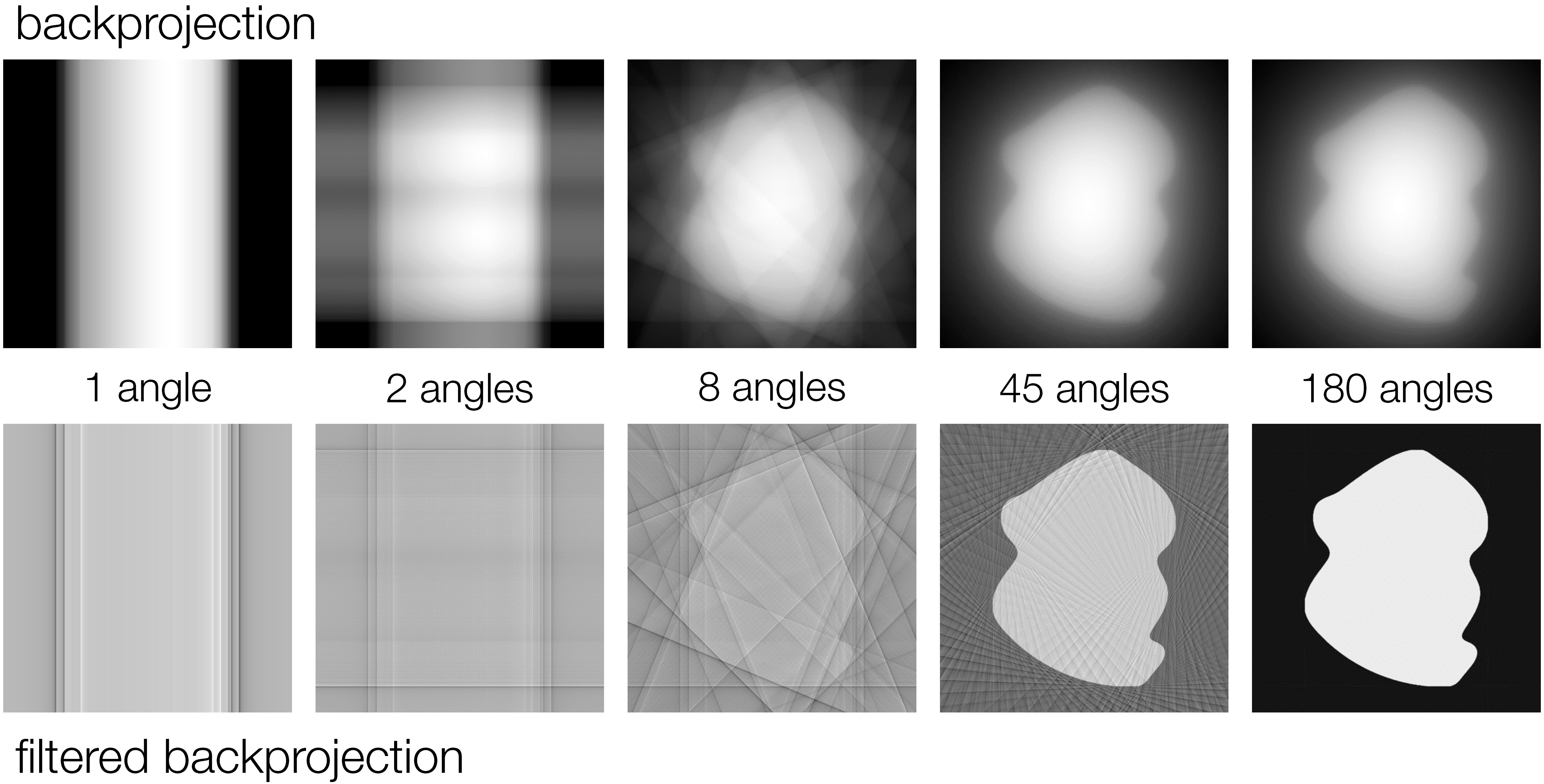
## Filtered Backprojection

$$f_{fbp}(x,y) = \int q_\theta(x\cos\theta + y\sin\theta)\, d\theta$$

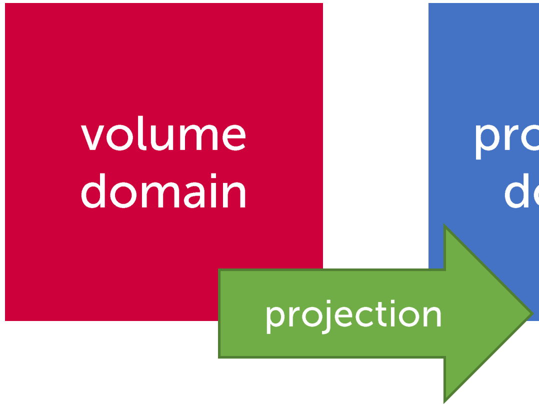
$$with \quad q_\theta(t) = \int P_\theta(\omega)|\omega|e^{i2\pi\omega t} d\omega$$

# Filtered Backprojection (FBP)



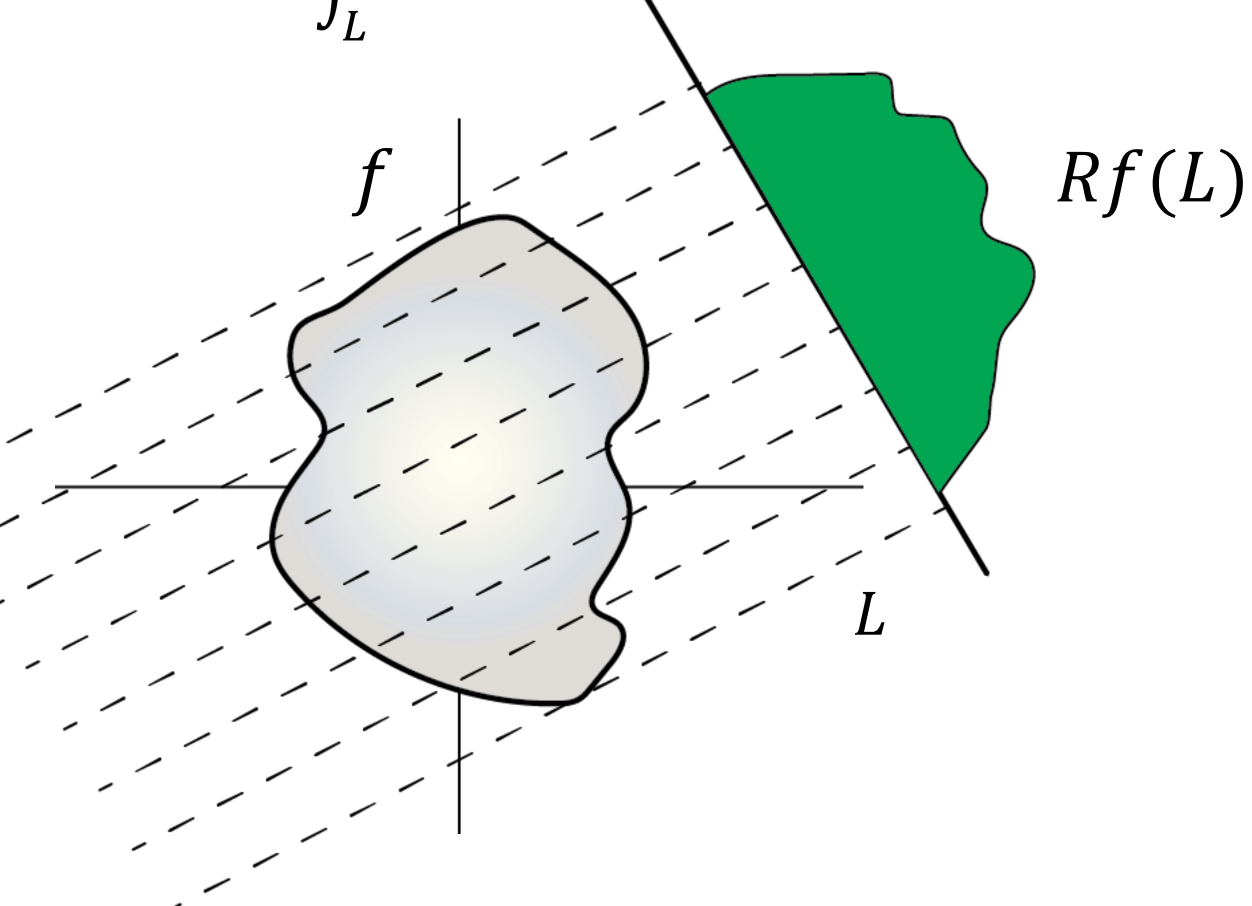


# Basic Mathematics of Computed Tomography



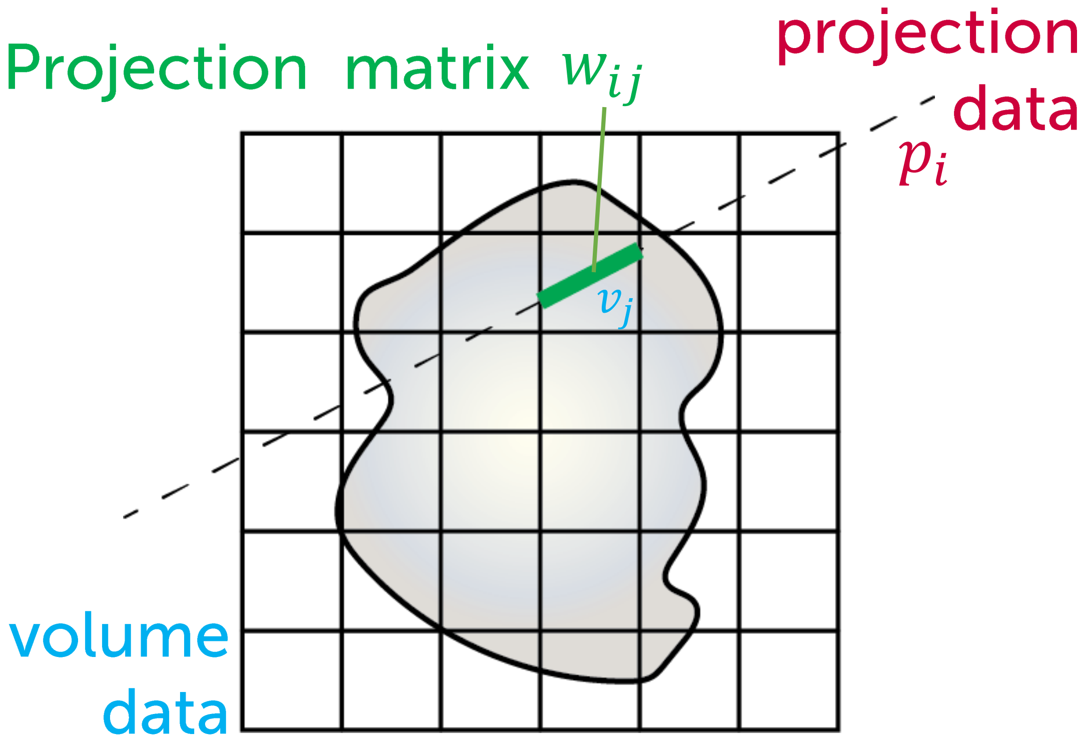

## analytically

$$Rf(L) = \int_L f(x)|dx|$$




## algebraically

$$p = Wv$$



# Algebraic Tomography

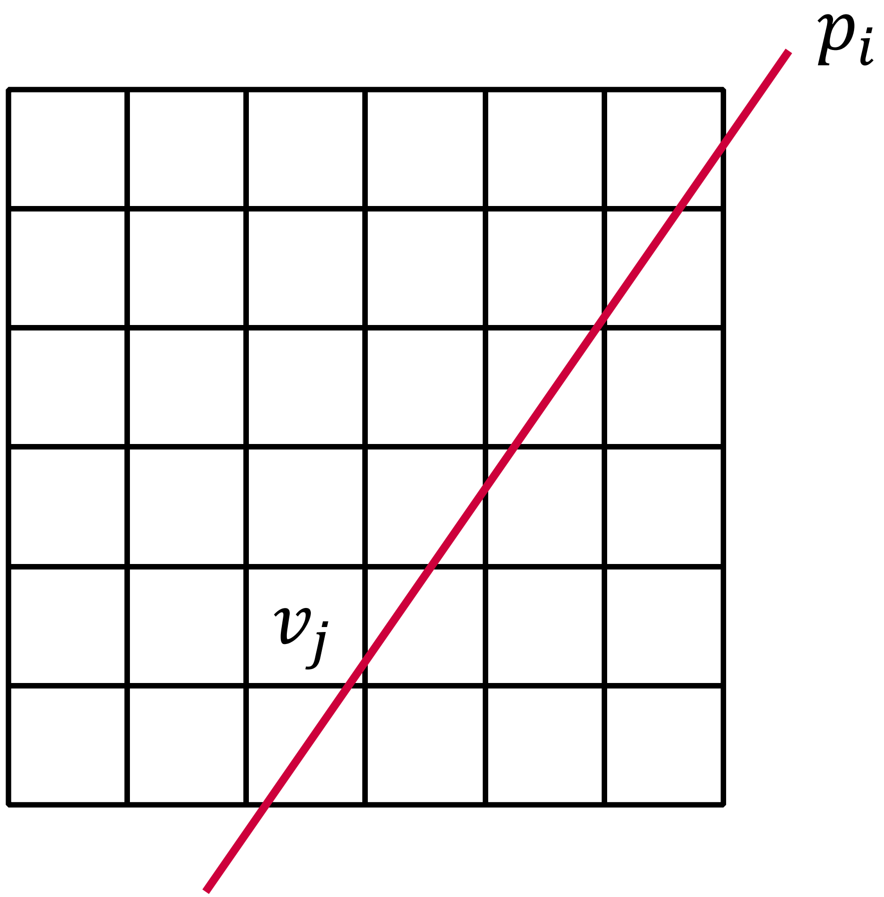
$$\boldsymbol{W}\boldsymbol{v} = \boldsymbol{p}$$

$\boldsymbol{v} \in \mathbb{R}^n$ Volume Vector

$\boldsymbol{p} \in \mathbb{R}^m$ Projection Data Vector

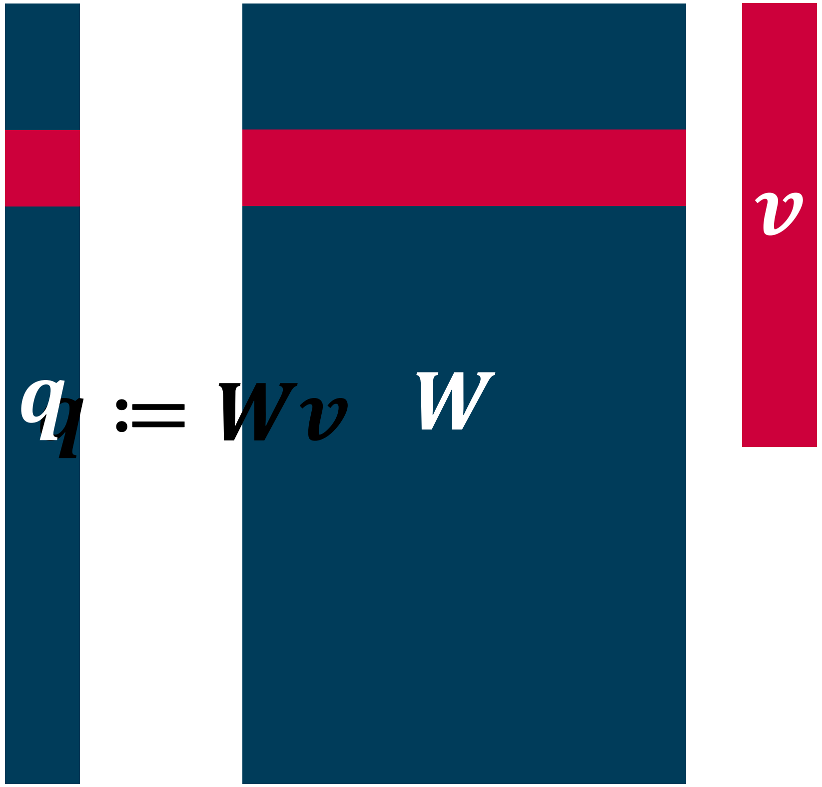
$\boldsymbol{W} \in \mathbb{R}^{m \times n}$ Projection Matrix

$w_{ij}$ represents how much $v_j$ contributes to $p_i$



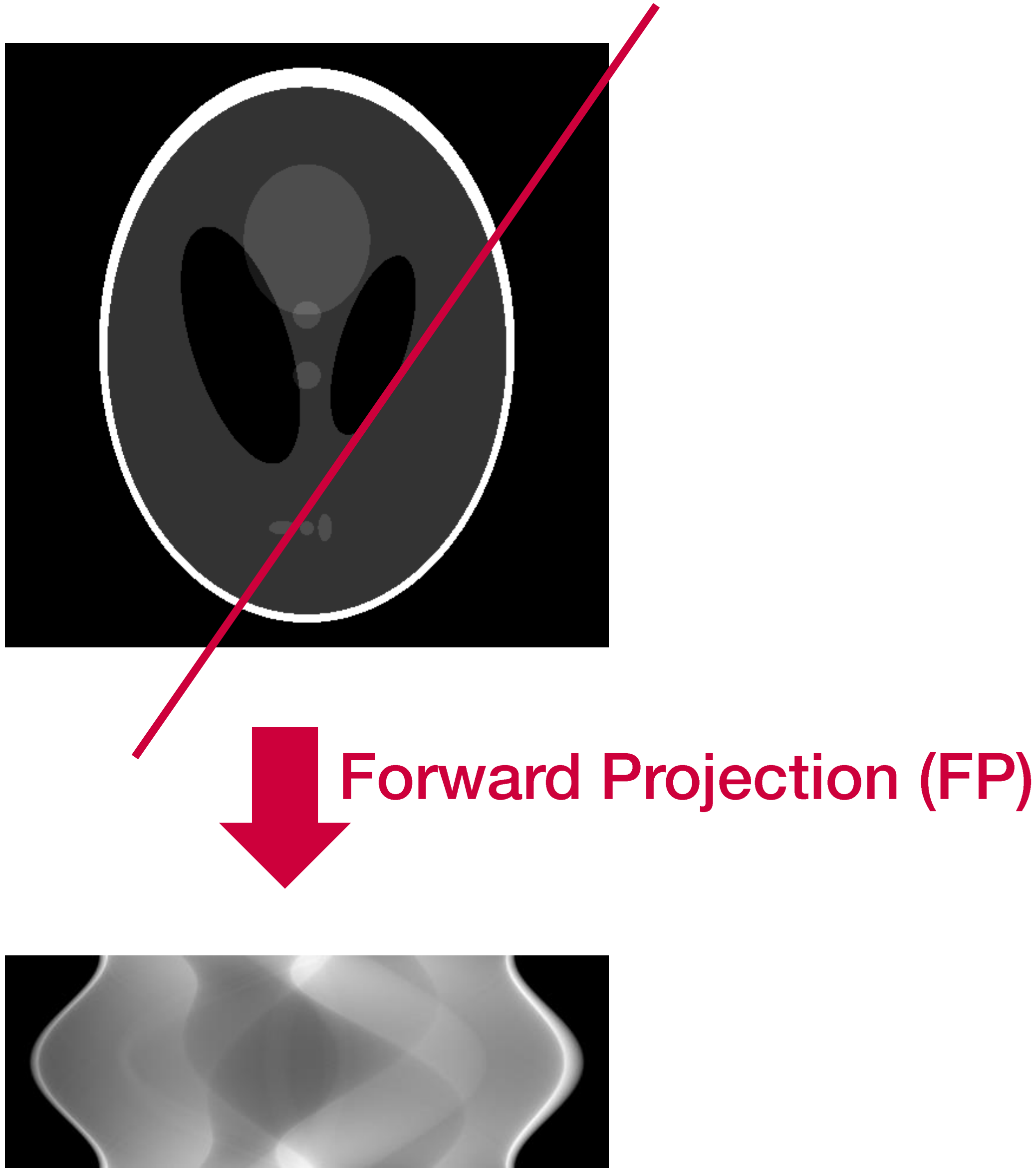
# Forward Projection






$$q := Wv$$

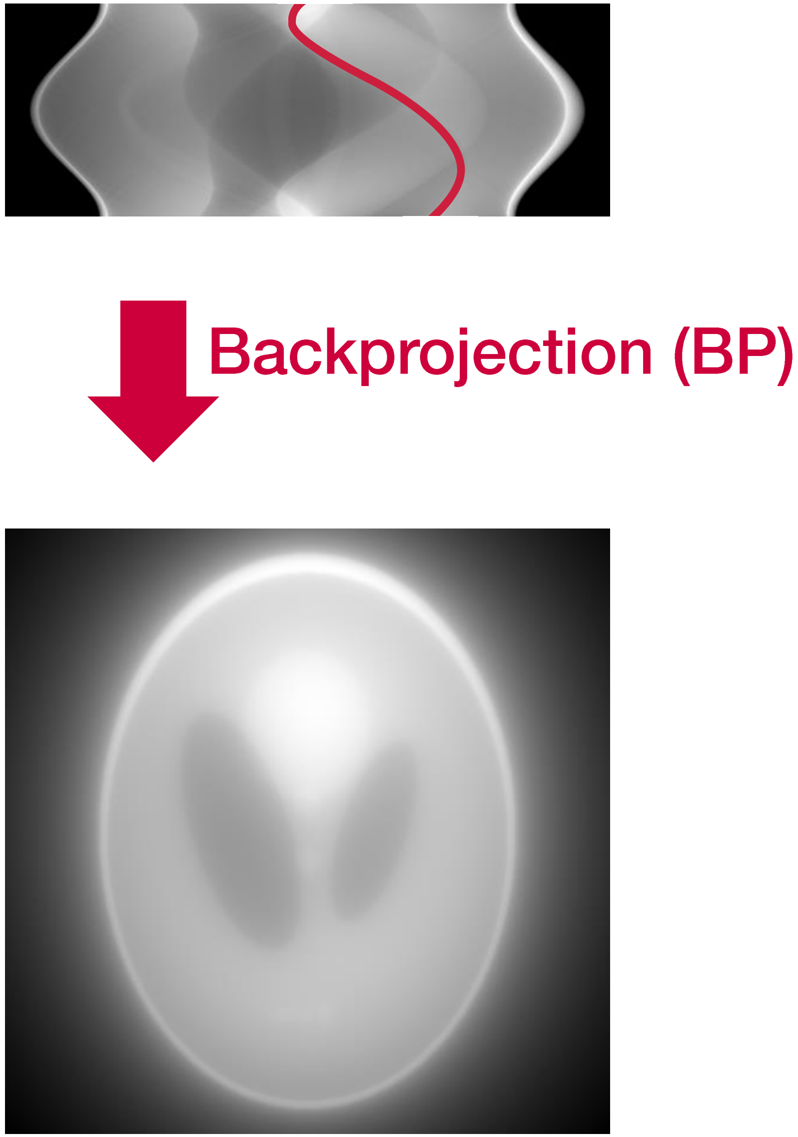
$$q_i := \sum_{j=0}^{n-1} w_{ij} v_j$$



# Backprojection

$$u := W^T p$$

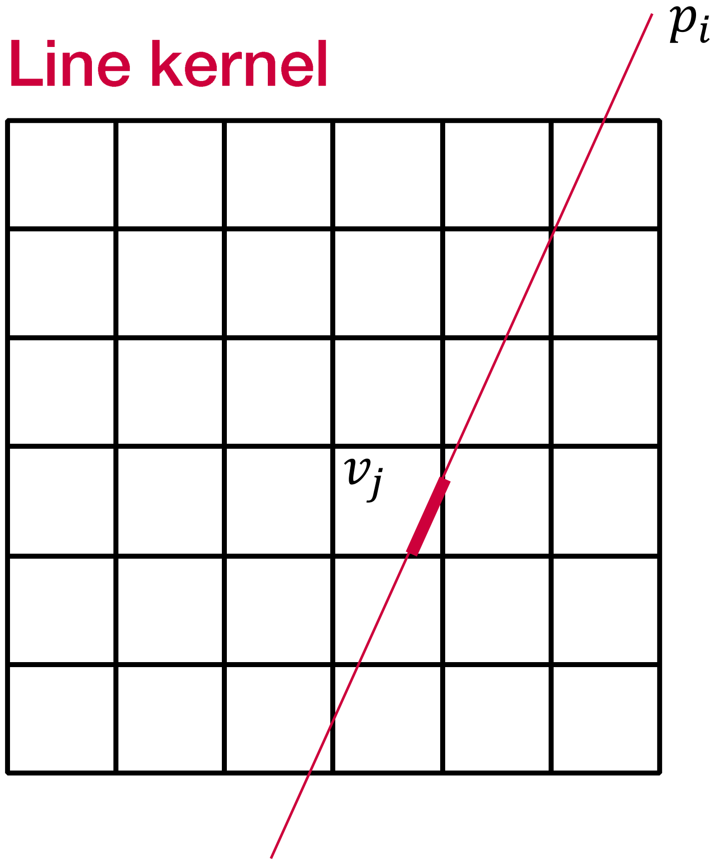
$$u_j := \sum_{i=0}^{m-1} w_{ij} p_i$$



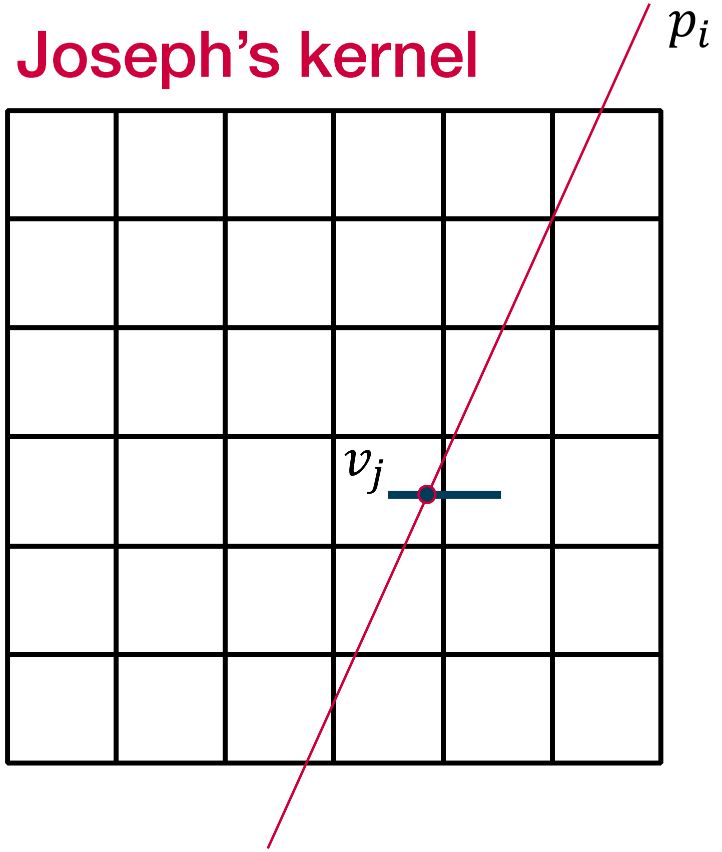
# Projection Matrix

$$\sum_{j=1}^{n} w_{ij} v_j = p_i$$

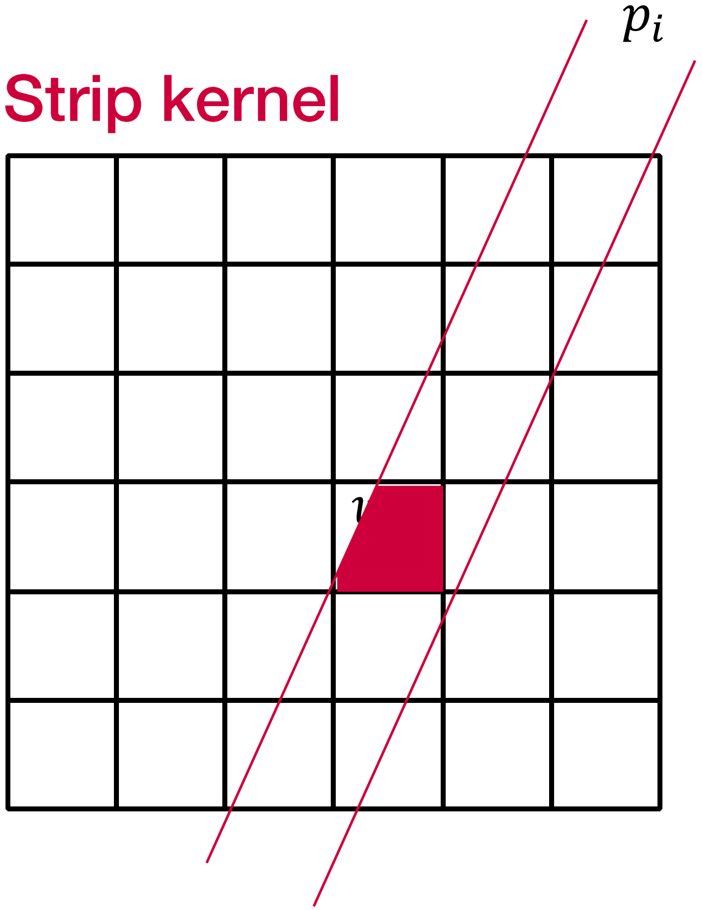
## Line kernel




## Joseph's kernel




## Strip kernel



# Algebraic Reconstruction

Solve $\boldsymbol{W}\boldsymbol{v} = \boldsymbol{p}$ → $\boldsymbol{v}^* = \boldsymbol{W}^{-1}\boldsymbol{p}$ (crossed out)

↓

$$\boldsymbol{v}^* = argmin_{\boldsymbol{v}} ||\boldsymbol{p} - \boldsymbol{W}\boldsymbol{v}||$$

some norm

Projection distance

Solve iteratively: Landweber (ART,SART,SIRT), Krylov Subspace (CGLS), …

# Algebraic Reconstruction

Universiteit Antwerpen

SIRT

Simultaneous Iterative Reconstruction Technique (SIRT)

Iteration step:

$$\boldsymbol{v}^{(k+1)} = \boldsymbol{v}^{(k)} + \boldsymbol{C}\boldsymbol{W}^T\boldsymbol{R}(\boldsymbol{p} - \boldsymbol{W}\boldsymbol{v}^{(k)})$$

$$\boldsymbol{R} \in \mathbb{R}^{m\times m} \quad r_{ii} = 1/\sum_{j=0}^{n-1} w_{ij}$$

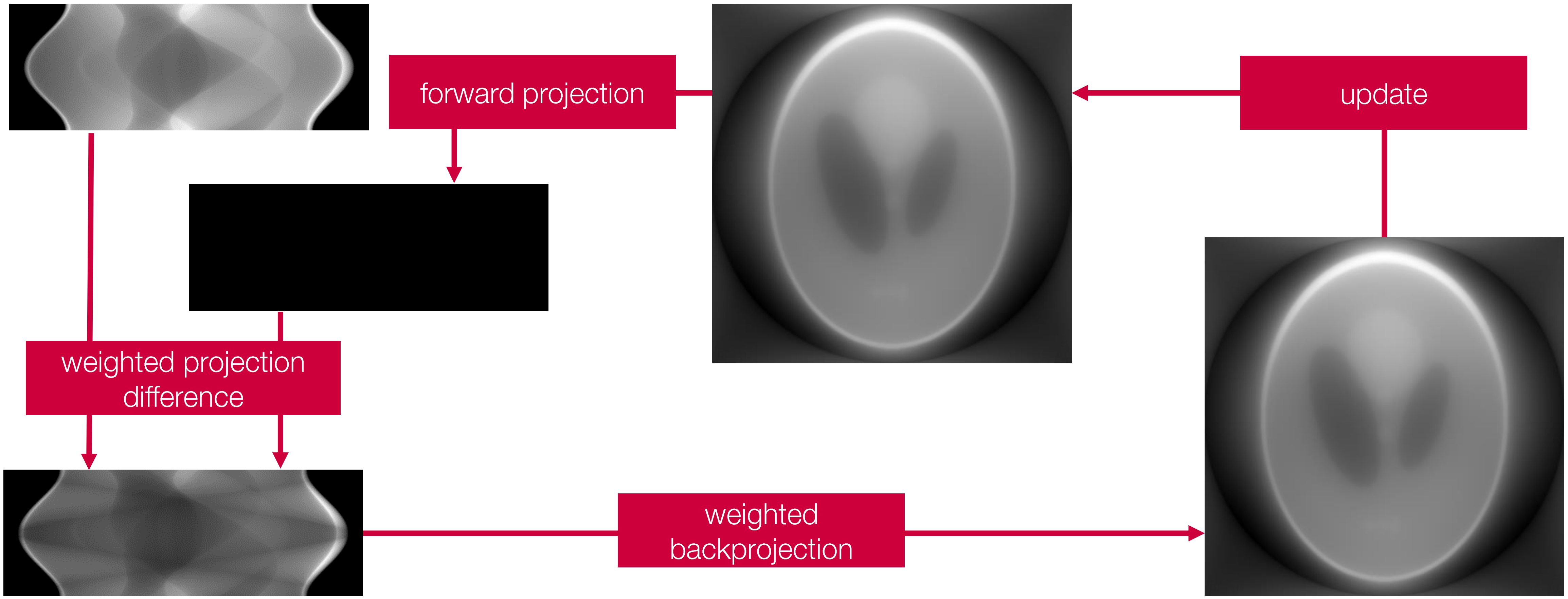
$$\boldsymbol{C} \in \mathbb{R}^{n\times n} \quad c_{jj} = 1/\sum_{i=0}^{m-1} w_{ij}$$

1. Weighted projection difference
2. Weighted backprojection
3. Update

Solves $\boldsymbol{v}^* = argmin_v ||\boldsymbol{p} - \boldsymbol{W}\boldsymbol{v}||_{\boldsymbol{R}}$ with $||\boldsymbol{x}||_R = \boldsymbol{x}^T\boldsymbol{R}\boldsymbol{x}$

# Algebraic Reconstruction

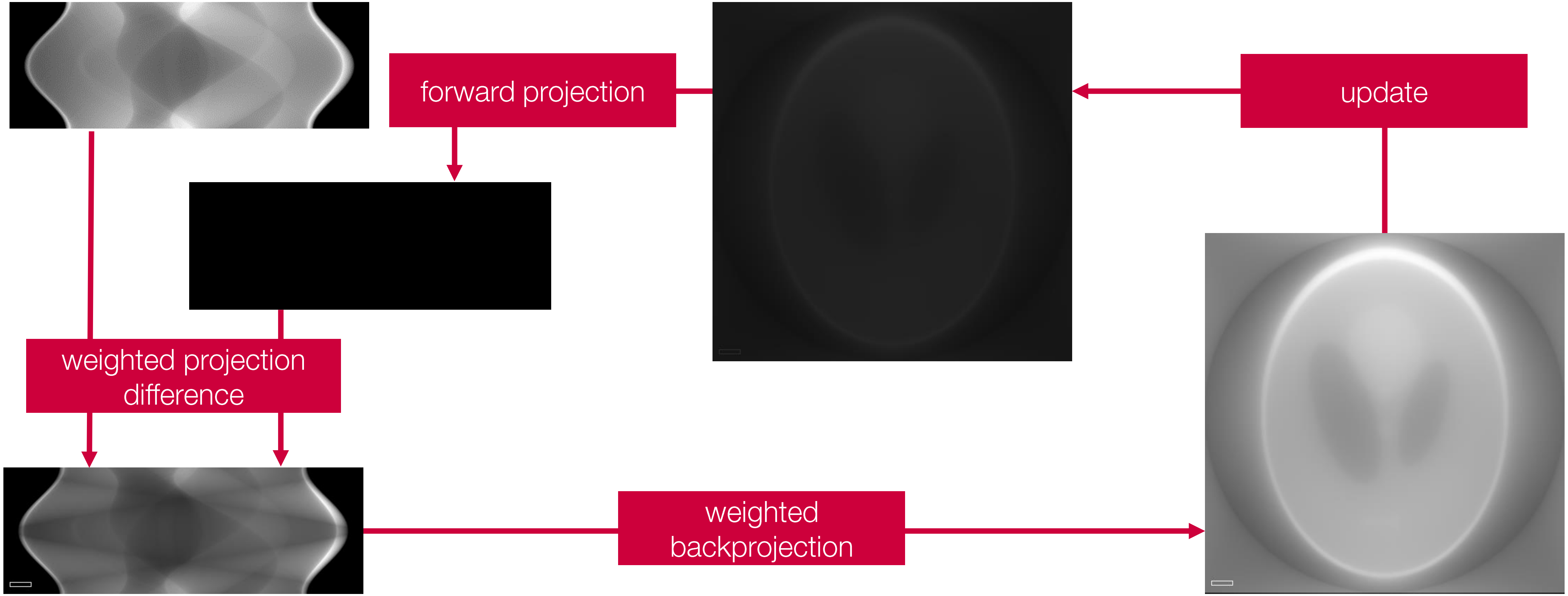
$$v^{(1)} = v^{(0)} + CW^{T}R(p - Wv^{(0)})$$



# Algebraic Reconstruction



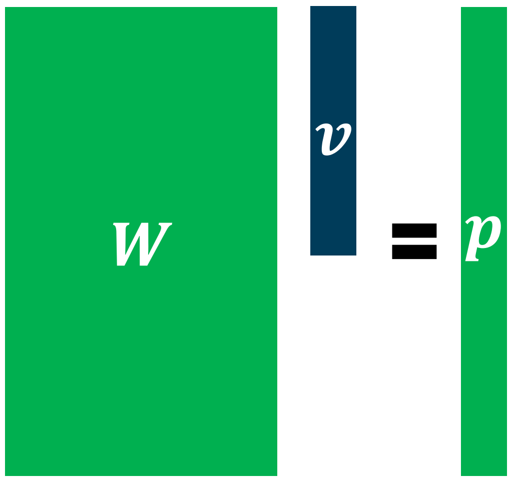
$$v^{(k+1)} = v^{(k)} + CW^T R(p - Wv^{(k)})$$



# Prior knowledge

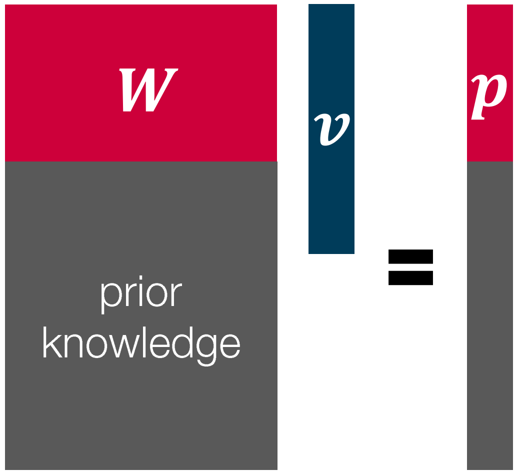


Sufficient projection data




Insufficient projection data


Insufficient projection data
+ prior knowledge



# Positivity constraint

Observation Negative values in the reconstruction make no sense.

Minimum Constraint: Project all negative values to zero.

$$\boldsymbol{v}^{(k+1)} = \wp\left(\boldsymbol{v}^{(k)} + \boldsymbol{C}\boldsymbol{W}^T\boldsymbol{R}\left(\boldsymbol{p} - \boldsymbol{W}\boldsymbol{v}^{(k)}\right)\right)$$

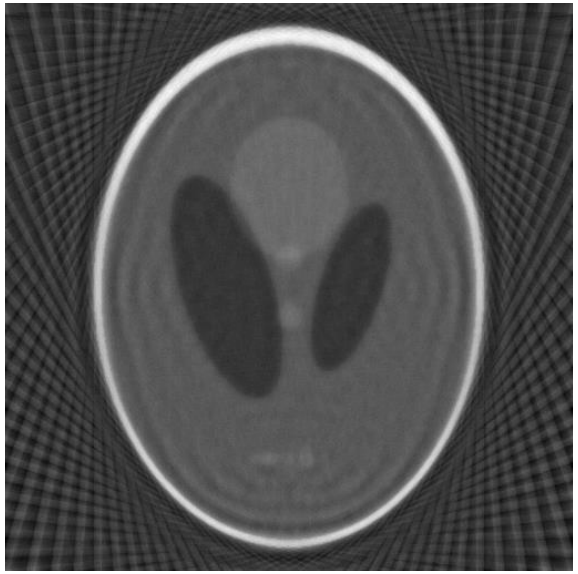
$$\wp(x_i) = \begin{cases} x_i & 0 \leq x_i \\ 0 & otherwise \end{cases}$$

Breaks mathematical converge theory of SIRT, but improves quality in practice.

# Positivity constraint

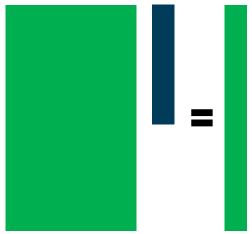


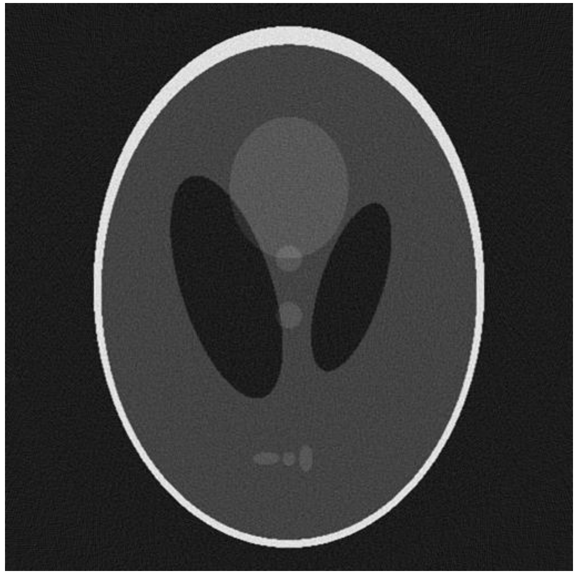
45 projection directions

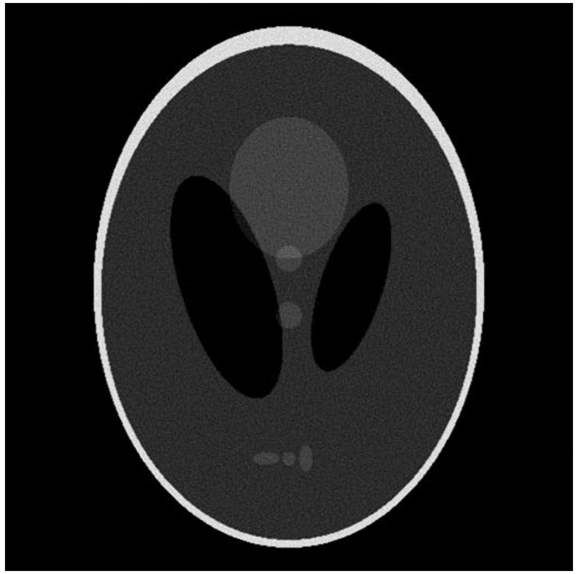


45 projection directions

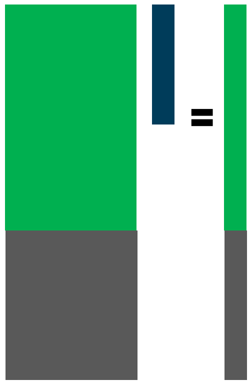
360 projection directions







360 projection directions

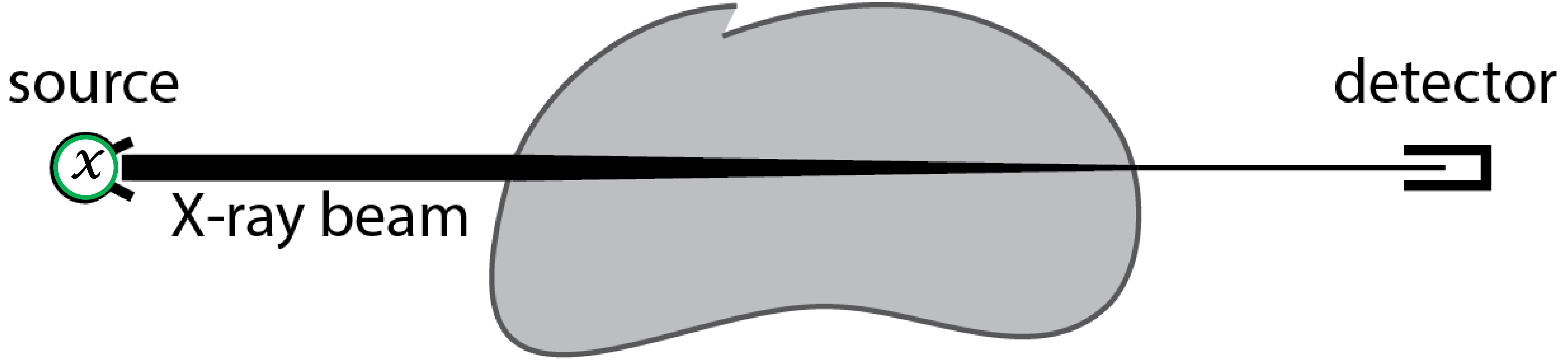


# Overview



- Computed Tomography
- Some Applications
- A Brief Introduction to X-ray Physics
- A Brief Introduction to Reconstruction Mathematics
- **When Physics and Mathematics colide**
  - **Polychromatic source and Beam Hardening**
  - **Scattering**

## X-ray detection




Beam intensity at source

$$I_0(E)$$

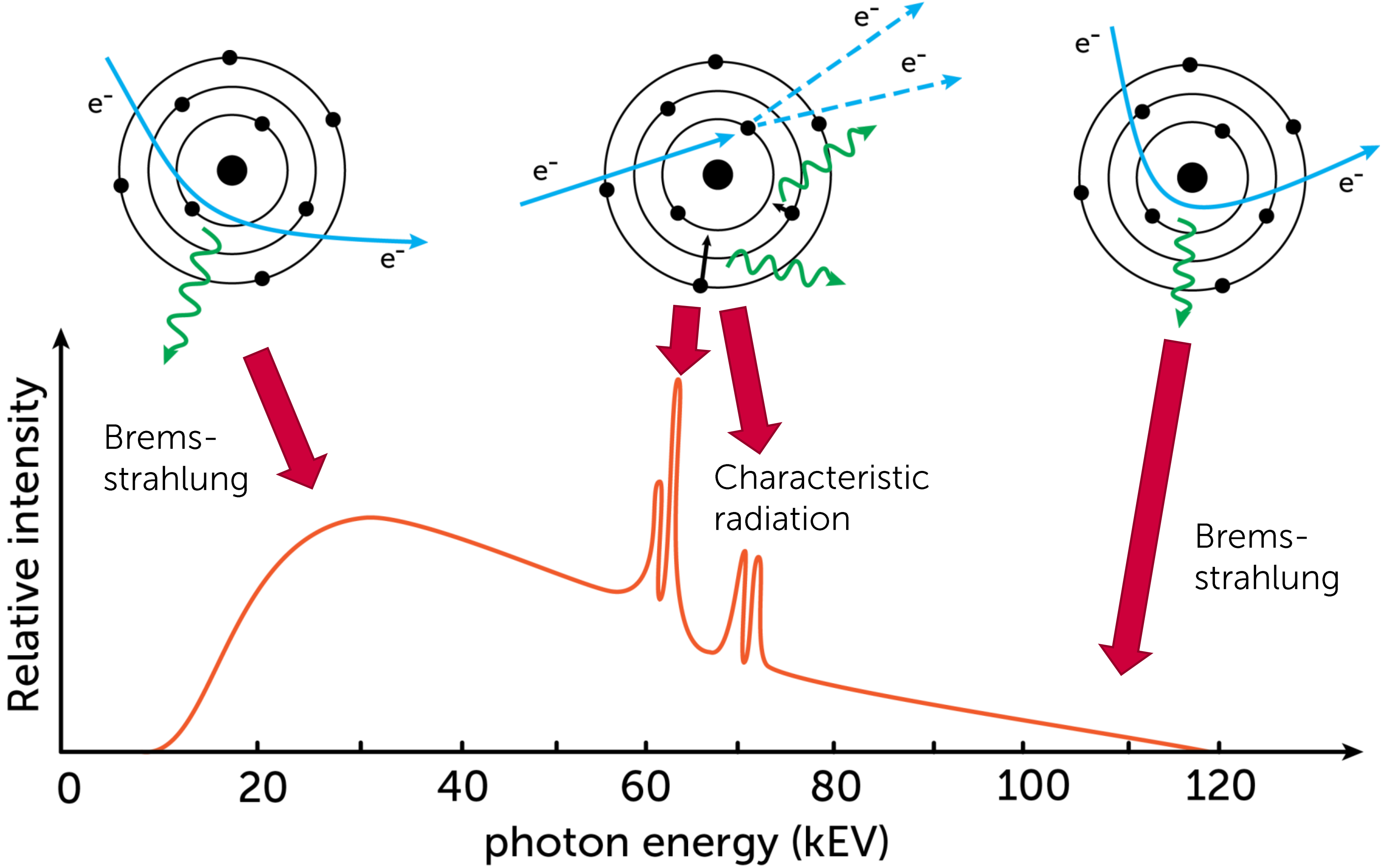
Beam intensity at detector

$$I(E) = I_0(E)e^{-\int \mu(\xi,E)d\xi}$$

Beam intensity changes

$$I(\xi + \Delta\xi, E) = I(\xi) - \mu(\xi, E)I(\xi, E)\Delta\xi$$

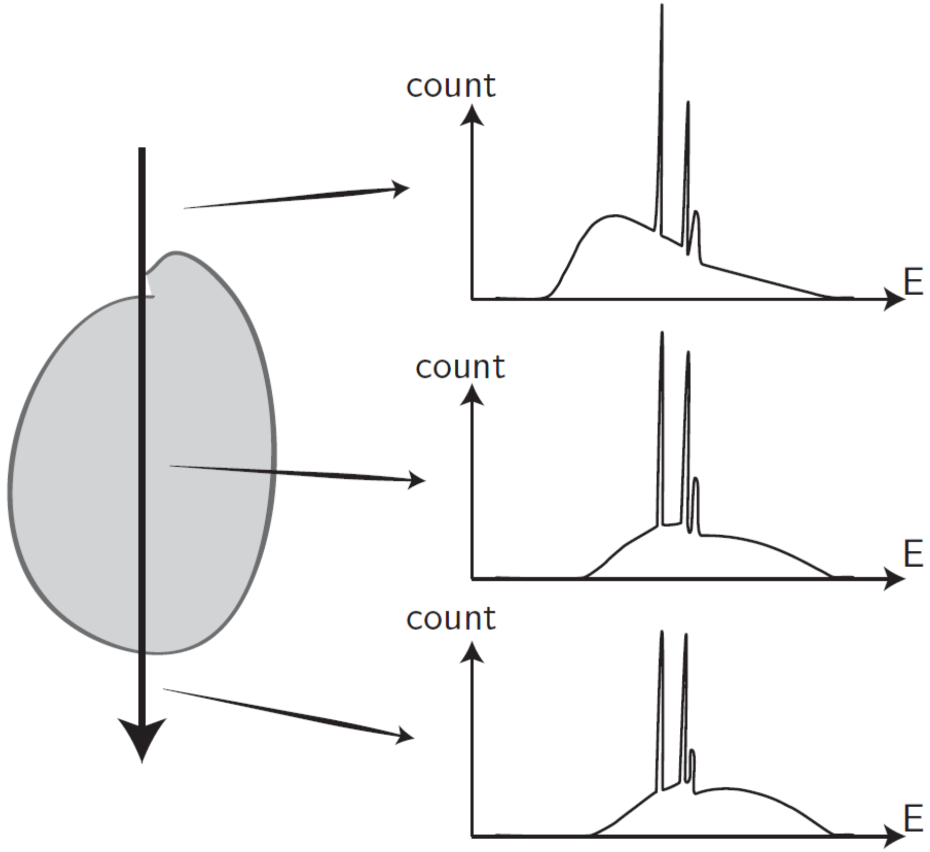
# X-ray spectrum



# Polychromatic spectrum

$$I_m(u,v) = \int_E I_{0,E}\, e^{-\int_L \mu(x,y,z,E)\, dL}\, dE$$

$$p_m(u,v) = -\log\left(\frac{I_m(u,v)}{I_0}\right)$$

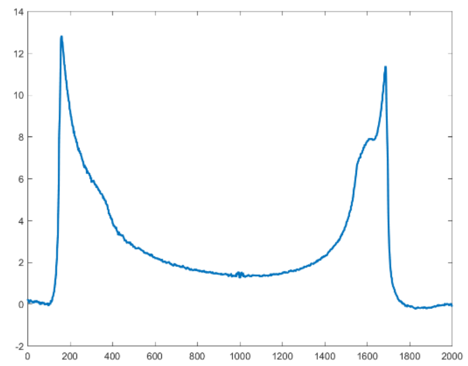


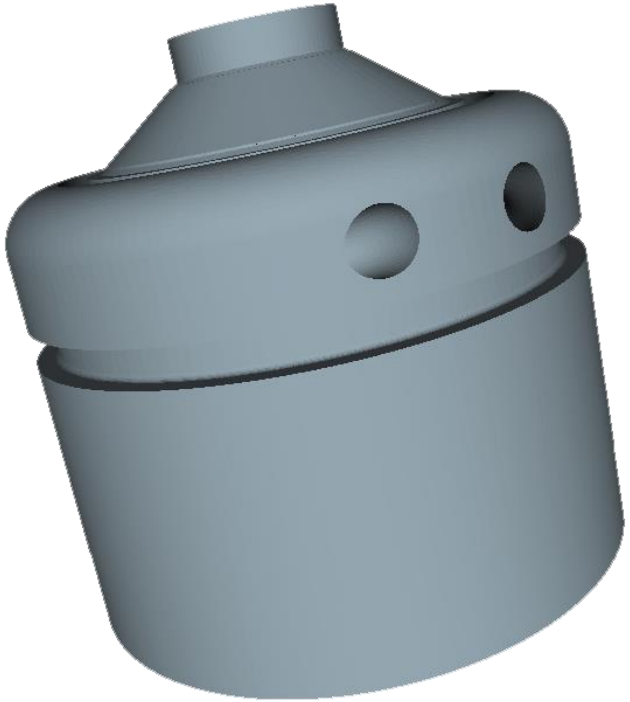

beam hardening

# Beam Hardening

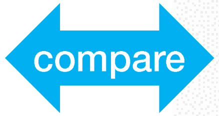


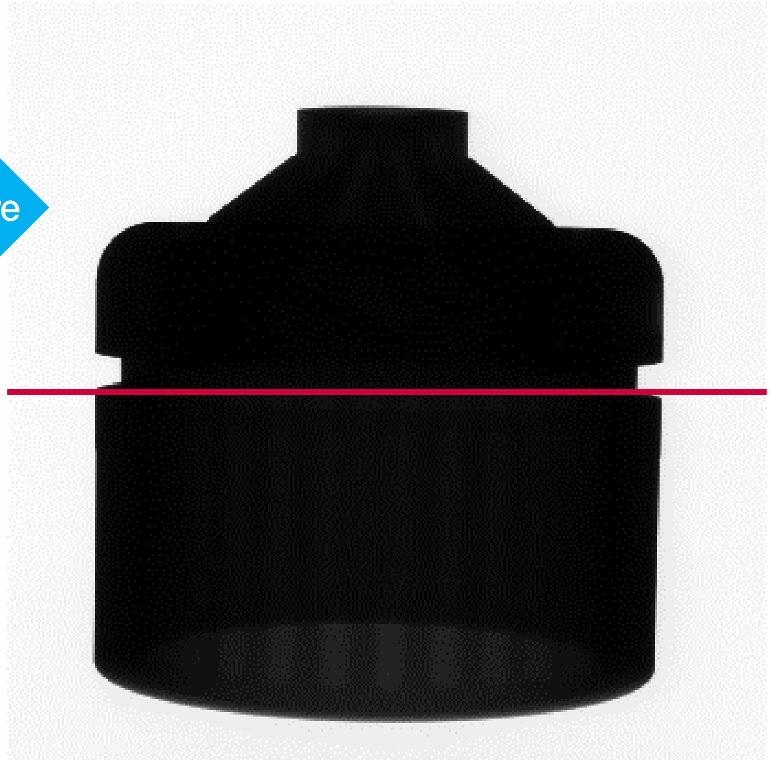
## injection mould, 3D printed



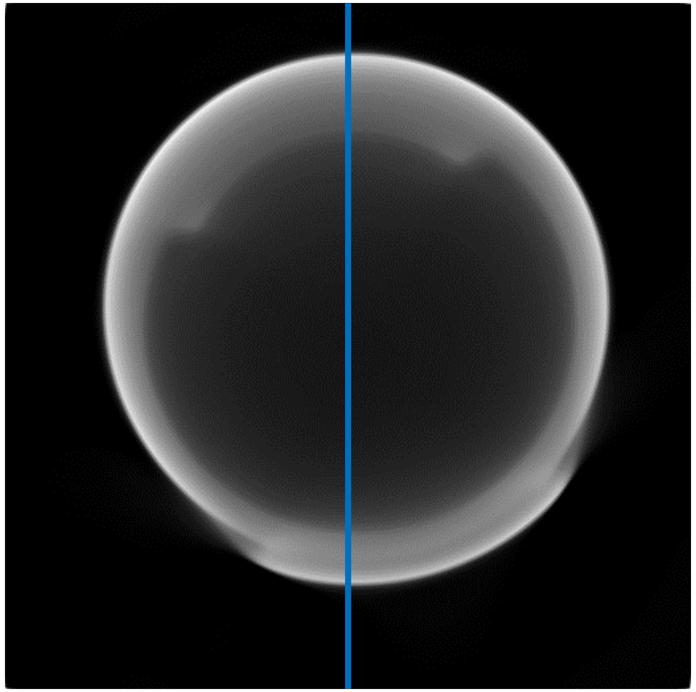



3D CAD model



projection



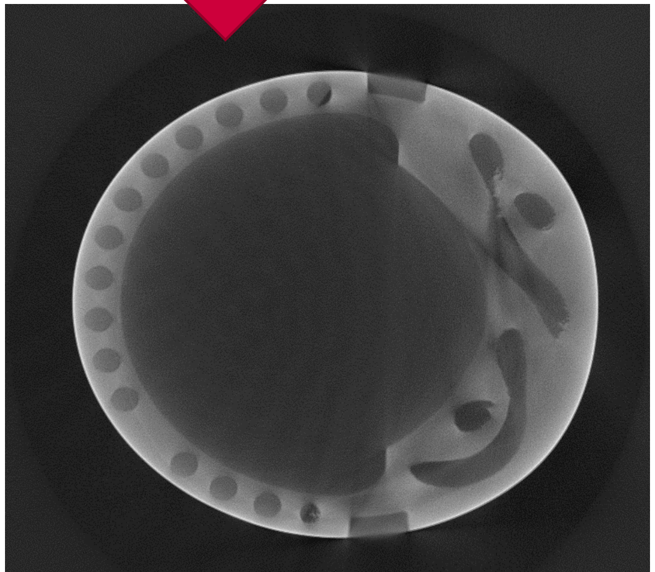
reconstruction

# Beam hardening correction strategy

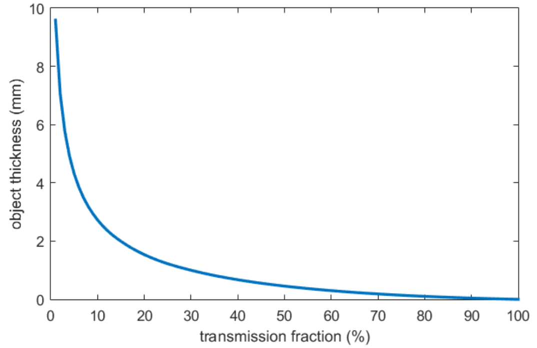


Transmission fraction (%) → Object thickness (mm)

log transform + FBP

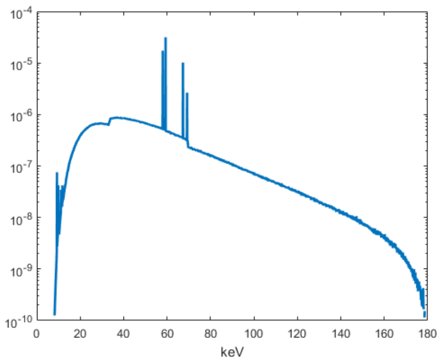


reconstructed intensities

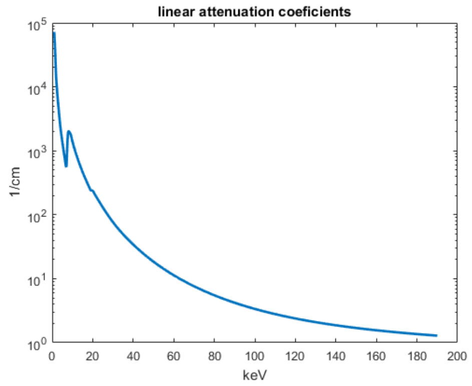



transmission vs thickness

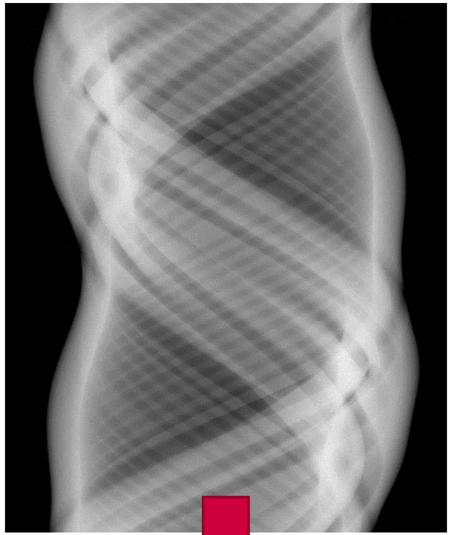
Monte Carlo simulation



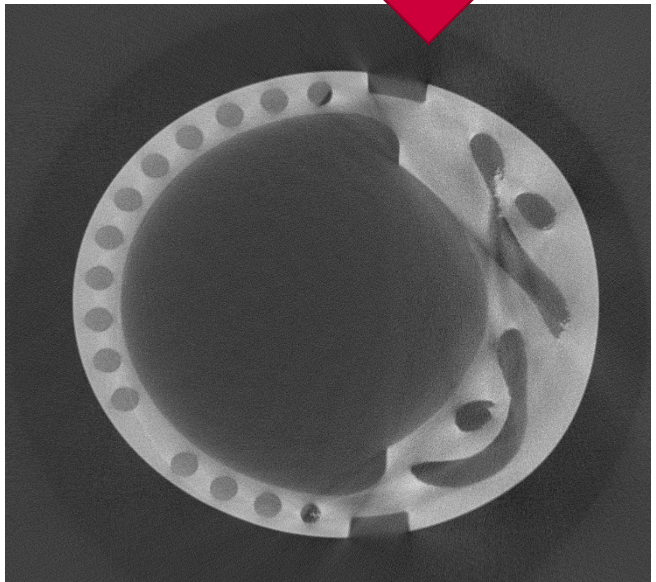

X-ray source spectrum




linear attenuation coefficients



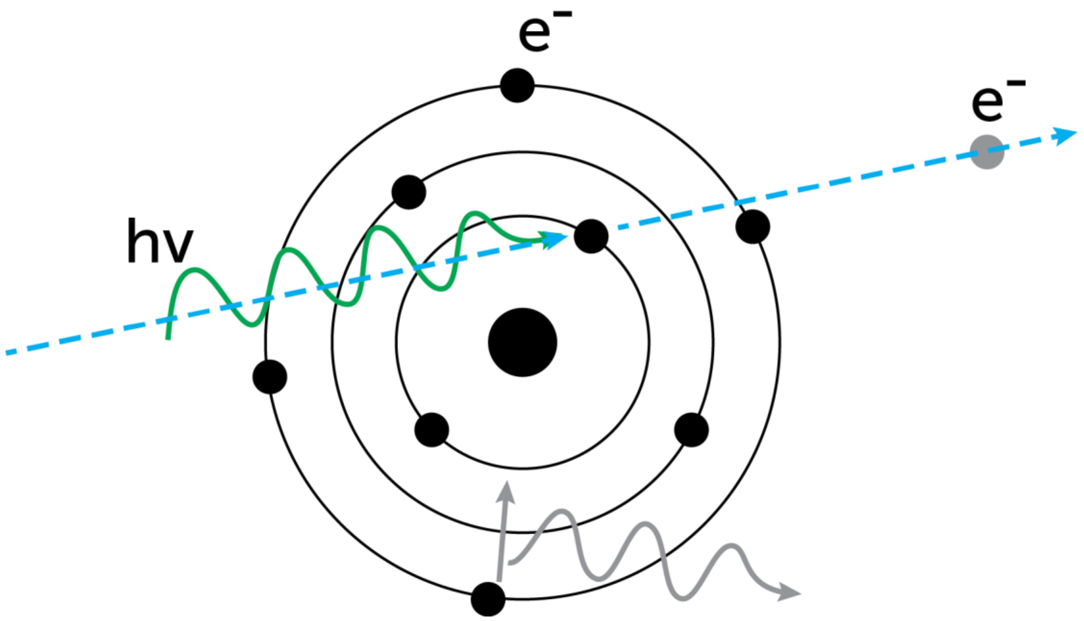
FBP



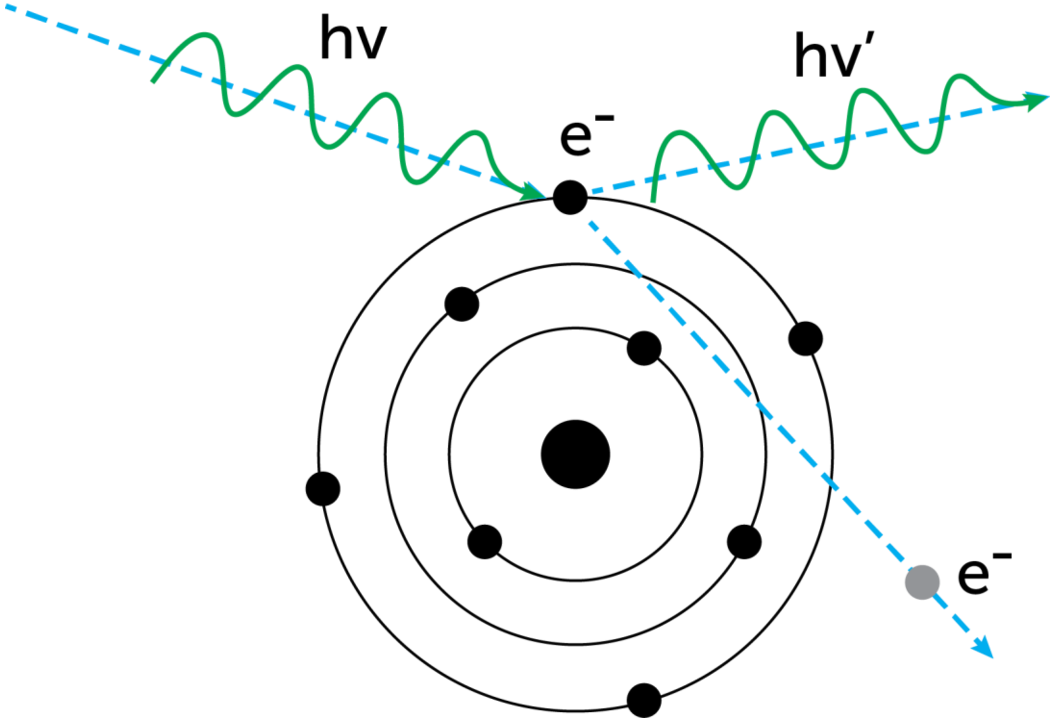
reconstructed line lengths

# X-ray Matter Interaction





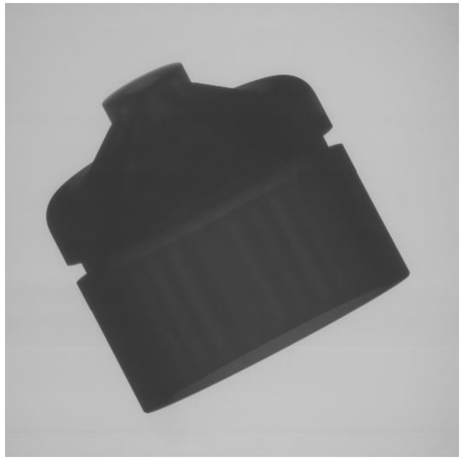

photoelectric effect




Compton scattering

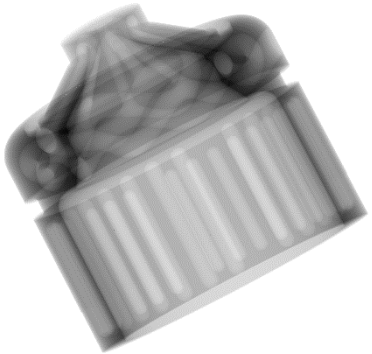
## Scattering

$$I_{S,D}(u,v) = I_0 e^{-\int_L \mu(x,y,z)\,dL} + I_0 e^{-\iint_L s(x,y,z)\mu(x,y,z)dL\,dxyz}$$



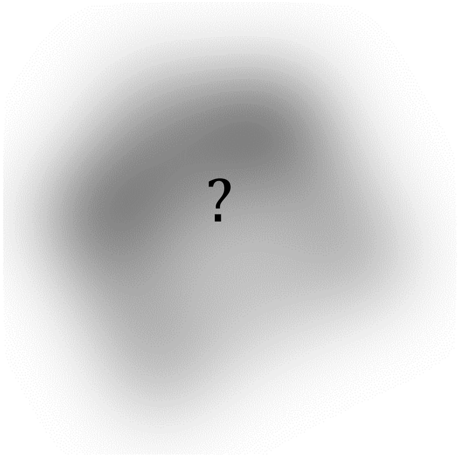
measured data

=



"clean" projection

+




scattering contribution

# Scatter correction strategy

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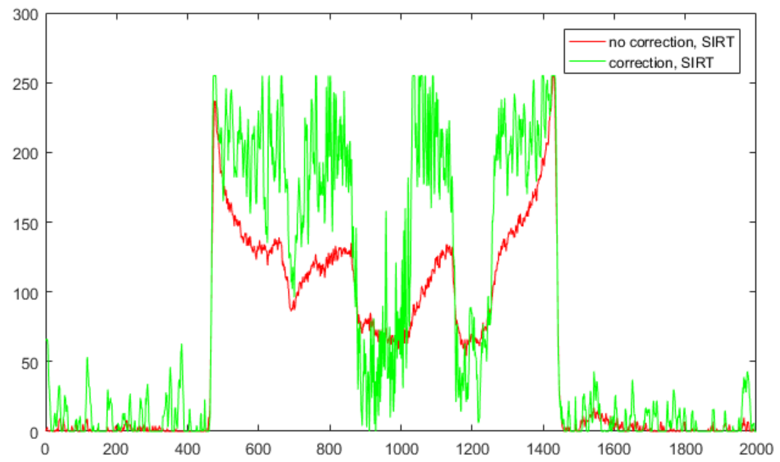
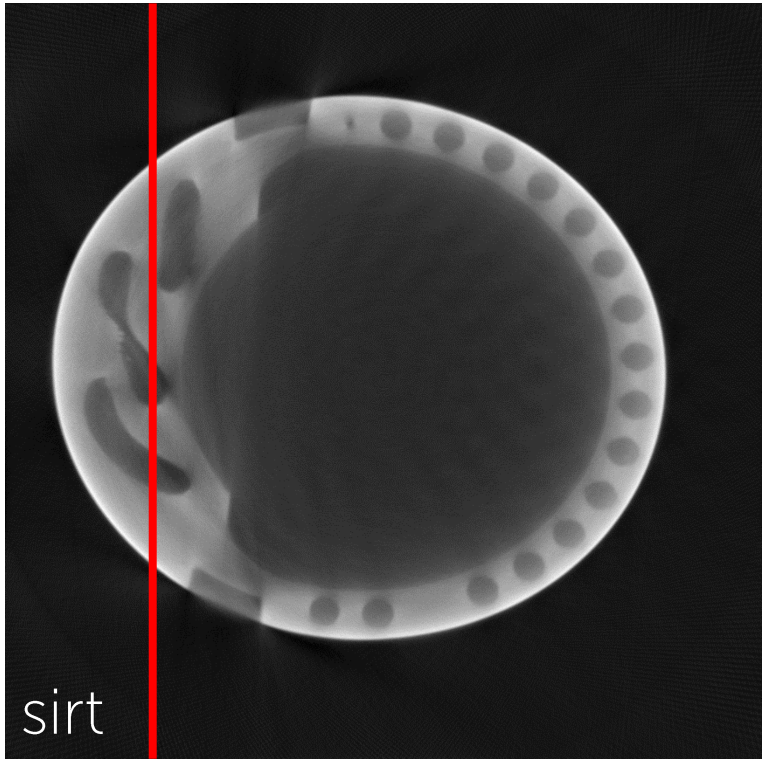
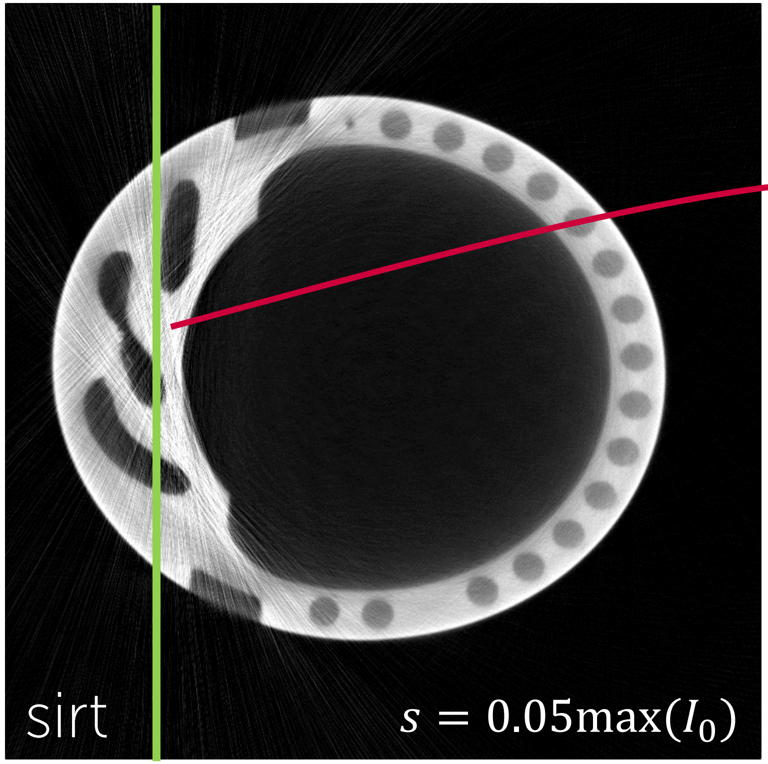
Simple approach: subtract constant value of all projections

$$-log\left(\frac{I-D}{F-D}\right)$$

$$-log\left(\frac{I-s-D}{F-D}\right)$$




noise!

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# Thanks!

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