Környezetfüggetlen rendtípusok

Iván Szabolcs
Roadmap

- Order types, well-orderings, ordinals and scattered order types
- Some applications of these constructs in computer science
- Order types of regular and context-free languages
- Known results regarding decidability and complexity issues
- Open questions of the area
- Some proof techniques
Ordered sets

A (linearly/totally) ordered set is a pair \((X, <)\) with \(X\) being a set and \(<\) being a total order on \(X\): an irreflexive, transitive and trichotomous relation.

In computer science we are usually only interested in countable sets since we want to represent their elements by a finite amount of information.

Examples

- The set \(\mathbb{N} = \{0, 1, 2, \ldots\}\) of natural numbers, equipped with their standard ordering.
- The set \(\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}\) of integers (+ their standard order).
- The set \(\{-4, -2, 0, 2, 4, \ldots\}\) of the even integers.
- The set \(\mathbb{Q}\) of rationals.
- The set \(\{0, 1\}^*\) of finite binary strings, ordered lexicographically.
Amongst these, there is an order-preserving bijection between the integers and the even integers.

This is an equivalence relation over all the possible orderings.

The classes of this equivalence relation are called order types.

The order type... 

- of the natural numbers is denoted by $\omega$
- of the integers (and of the even integers) is denoted by $\zeta$
- of the rationals is denoted by $\eta$
- of the finite sets is denoted by their cardinality

  e.g. the order type sun < mon < tue < ... < sat of the days of the week is denoted by 7

these order types are pairwise different, e.g. $\omega \neq \zeta$
If \((X, <)\) and \((Y, \prec)\) are ordered sets of order type \(o_X\) and \(o_Y\), respectively, then the order type of their (disjoint) union \(X \times \{0\} \cup Y \times \{1\}\), ordered by

- each element of \(X\) is smaller than each element of \(Y\),
- inside \(X\) and \(Y\), the elements are ordered according to the original \(<\) and \(\prec\), resp,

is usually denoted by \(o_X + o_Y\).

Two copies of the natural numbers, placed next to each other:

\[(0, 0) < (1, 0) < (2, 0) < \ldots < (0, 1) < (1, 1) < (2, 1) < \ldots\]

has order type \(\omega + \omega\).

\[
\omega + \omega \neq \omega \quad \quad \quad \eta + \eta = \eta
\]
\[
\omega + 1 \neq \omega \quad \quad \quad 1 + \omega = \omega
\]
If \( I = (I, <) \) is an “indexing” ordering and for each \( i \in I \), \( X_i = (X_i, <_i) \) is an ordering, then \( \sum_{i \in I} X_i \) is the ordering with:

- domain \( \bigcup_{i \in I} X_i \times \{i\} \)
- equipped with the anti-lexicographic ordering: \( (p, i) < (q, j) \) if and only if either \( i < j \) or \( (i = j \text{ and } p <_i q) \)

The order type is denoted \( \sum_{i \in I} o(X_i) \).

We can place a number of orderings, each being either of type 1 or of type \( \omega \), next to each other, indexed by \( \omega \) and we can get e.g.:

- \( 1 + 1 + 1 + 1 + 1 + \ldots = \omega \)
- \( 1 + 1 + 1 + 1 + \omega + 1 + 1 + 1 + 1 + \ldots = \omega + \omega \)
- \( 1 + \omega + 1 + 1 + \omega + 1 + 1 + 1 + \omega + \ldots = \omega + \omega + \omega + \ldots \)
If we have a sum of the form $\sum_{i \in I} X_i$ with each $X_i$ having the same order type $o$, then the order type of this sum is also denoted by $o \times o(I)$.

\[
\omega + \omega + \omega + \ldots = \omega \times \omega \\
\omega + 1 + \omega + 1 + \omega + 1 + \ldots = (\omega + 1) \times \omega = \omega \times \omega \\
\omega + \omega = \omega \times 2 \\
2 \times \omega = 2 + 2 + 2 + \ldots = \omega \\
\ldots + \omega + \omega + \omega + \ldots = \omega \times \zeta \\
\zeta + \zeta + \zeta + \ldots = \zeta \times \omega \neq \omega \times \zeta
\]
Well-orderings, ordinals

- An ordering is a **well-ordering** if it contains no infinite descending chain.
- The order types of well-orderings are called **ordinals**.

- The finite order types $0$, $1$, $2$, ... are ordinals, as well as $\omega$.
- $\zeta$ and $\eta$ are **not** ordinals.
- $\omega \times \omega$ is an ordinal.
The order of the ordinals

• the (countable) ordinals themselves are also ordered: \( o_1 \leq o_2 \) if some ordering of type \( o_1 \) can be mapped into an ordering of type \( o_2 \) in an order-preserving way
• if not the other way around: \( o_1 \prec o_2 \)
  • \( \omega \prec \omega + 1 \)
  • \( \omega + \omega \prec \omega \times \omega \)
• turns out \( \prec \) is a total ordering over the ordinals: for each pair \( o_1, o_2 \) of ordinals, exactly one of \( o_1 \prec o_2, o_2 \prec o_1 \) or \( o_1 = o_2 \) holds
• (this is not true for all the order types, e.g. \( \eta + 1 + 1 + \eta \) can be embedded into \( \eta \) and vice versa but they are not the same)
• moreover, this \( \prec \) contains no infinite descending chains \( \Rightarrow \) the ordinals themselves are also well-ordered
Successor and limit ordinals

An ordinal $\alpha$ is either...

- a **successor** ordinal, that is, $\alpha = \beta + 1$ for some (smaller) ordinal $\beta$,
- or a **limit** ordinal, that is, $\alpha = \bigvee_{\beta < \alpha} \beta$ is the supremum of all the ordinals smaller than $\alpha$

- $7 = 6 + 1$ is a successor ordinal
- $\omega$ is a limit ordinal: it is the supremum of $\{0, 1, 2, 3, \ldots \}$
- $\omega + 3 = (\omega + 2) + 1$ is a successor ordinal
- $\omega + \omega$ and $\omega \times \omega$ are limit ordinals
- $0 = \bigvee \emptyset$ is a limit ordinal  
  (usually treated separately in proofs)
Exponentation of ordinals

If $\alpha$ and $\beta$ are ordinals, then $\alpha^\beta$ is...

- 1 if $\beta = 0$,
- $(\alpha^\gamma) \times \alpha$ if $\beta = \gamma + 1$ is a successor ordinal,
- 0 if $\beta$ is a limit ordinal and $\alpha = 0$,
- $\bigvee_{\gamma < \beta} \alpha^\gamma$ if $\beta$ is a limit ordinal and $\alpha \neq 0$.

- $\omega^1 = (\omega^0) \times \omega = 1 \times \omega = \omega$
- $\omega^2 = (\omega^1) \times \omega = \omega \times \omega$
- $\omega^3 = \omega \times \omega \times \omega$
- $\omega^\omega = \bigvee_{n < \omega} \omega^n = 1 + \omega + \omega^2 + \omega^3 + \ldots$

associative
Cantor normal form

Each ordinal $\alpha$ can be uniquely written as a finite sum of the form

$$\alpha = \omega^{\alpha_1} \times n_1 + \omega^{\alpha_2} \times n_2 + \ldots + \omega^{\alpha_k} \times n_k$$

for some integer $k \geq 0$, ordinals $\alpha_1 > \alpha_2 > \ldots > \alpha_k$ and integer coefficients $n_1, \ldots, n_k > 0$.

- $\omega^2 + \omega^2 + \omega + \omega + \omega + 2 = \omega^2 \times 2 + \omega \times 3 + 2$
- $\omega + \omega^2 + \omega + \omega^2 + \omega = \omega^2 \times 2 + \omega$
- $\omega^\omega \times (\omega + 1) = \omega^{\omega+1} + \omega^\omega$

seems like a finitely presentable normal form for ordinals

$$\epsilon_0 = 1 + \omega + \omega^\omega + \omega^{\omega^\omega} + \omega^{\omega^{\omega^\omega}} + \ldots$$

Then, $\epsilon_0 = \omega^{\epsilon_0}$.  

this guy is still countable
Applications of ordinals

Halting conditions

Since the ordinals themselves are well-ordered, if we can assign an ordinal to each program state such that the ordinal decreases in each step, we proved termination.

Ackermann function

\[
A(n, m) := \begin{cases} 
  m + 1 & \text{if } n = 0 \\
  A(n - 1, 1) & \text{if } n > 0 \text{ and } m = 0 \\
  A(n - 1, A(n, m - 1)) & \text{if } n > 0 \text{ and } m > 0
\end{cases}
\]

If we assign \( \omega \times n + m \) to each recursive call, then:

- first case: instant termination
- second case: \( \omega \times n + 0 \geq \omega \times (n - 1) + 1 \)
- third one: \( \omega \times n + m - 1 < \omega \times n + m \) so the inner call terminates by induction and becomes some finite number \( M \), and \( \omega \times (n - 1) + M < \omega \times n + m \) tools exist to assign ordinals automatically to functions.
Hercules vs the Hydra

Kirby-Paris, 1982
Hercules vs the Hydra

Kirby-Paris, 1982
Hercules vs the Hydra
Hercules vs the Hydra
Hercules vs the Hydra
Hercules vs the Hydra
Ordinal of the Hydra

Let us assign an ordinal to each Hydra as follows:

- the single-point hydra’s ordinal is 0
- if the children of the Hydra have the ordinals $\alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_n$, then the Hydra gets the ordinal $\omega^{\alpha_1} + \omega^{\alpha_2} + \ldots + \omega^{\alpha_n}$

\[
\omega^0 + \omega^0 = 1 + 1 = 2
\]

\[
\omega^2 \times 2
\]
Ordinal of the Hydra

\[ \omega^\omega \times 4 \]

\[ \omega + 2 \]

\[ \omega + 2 \]

\[ \omega + 2 \]

\[ \omega + 2 \]

\[ \omega + 2 \]
The ordinal of the Hydra always decreases.
The ordinal of the Hydra always decreases

Hercules always wins, no matter what
How to represent order types by a finite description?

Ordinals smaller than $\epsilon_0$ can be represented by a recursive Cantor normal form.

An idea: let us use lexicographic orderings of formal languages!

- binary words, say $u < v$ iff either $u$ is a prefix of $v$, or $u = x0y$, $v = x1z$ for some $x, y, z$

- $\varepsilon < 0 < 00 < 000 < 0000 < \ldots$, so $o(0^*) = \omega$
- $\ldots < 0001 < 001 < 01 < 1$, so $o(0*1) = -\omega$, the order type of the negative integers
- $o(0^*(0*1 + 1^+)) = \zeta$
- $10 < 100 < 1000 < \ldots < 110 < 1100 < 11000 < \ldots < 1110 < \ldots$, so $o(1^+0^+) = \omega^2$
- $o((00 + 11)^*01) = \eta$
What order types can be represented?

Every countable order type is the (lexicographic) order type of some language over \{0, 1\}.

Main questions

- How can we define a language?
  - regular languages
  - context-free languages
  - context-sensitive languages
  - one-counter languages

- Can we work by order types given by languages at all?
  - **Isomorphism problem**: can we decide for two languages \(K\) and \(L\) whether \(o(K) = o(L)\)?
  - Can we “compute” \(o(KL), o(K \cup L)\) or \(o(K^*)\) if we know \(K, L, o(K)\) and \(o(L)\) in some other representation? say, their Cantor normal form if they are ordinals
Well-orderings do not contain $-\omega$.

Scattered orderings are those not containing $\eta$. Their order types are the scattered order types.

- Each ordinal is scattered.
- $\zeta$ is scattered.
- $\zeta \times \zeta$ is scattered.
- $\omega + (\omega) + (\omega) + \omega + (\omega) + \omega + \omega + (\omega) + \ldots$ is scattered.
  
  there are already uncountably many from these guys

- $\{0, 1\}^*$ is not scattered.
Hausdorff’s theorem

Hausdorff assigned to each scattered (countable) order type a (countable) ordinal, its “rank” (intuitively, a sort of “embedding depth”).

To each ordinal \( \alpha \), let us define a class \( H_\alpha \) of orderings as follows:

- let \( H_0 \) contain all the finite orderings; this is an Ésik-Iván modification from 2012
- for \( \alpha > 0 \), let \( H_\alpha \) be the smallest class of orderings that is
  - closed under finite sum and
  - contains all the orderings of the form \( \sum_{i \in \zeta} X_i \) with each \( X_i \) being in some \( H_{\alpha_i} \) with \( \alpha_i < \alpha \).

If an ordering is a member of an \( H_\alpha \), let its rank be the smallest such \( \alpha \).

rank of an order type is defined in the expected way.
Rank examples

- 0 and 1 are finite, so they have rank 0.
- \( \omega = \ldots + 0 + 0 + 0 + 1 + 1 + 1 + 1 + 1 + \ldots \) is a \( \zeta \)-sum of order types of rank smaller than 1, so its rank is 1.
- \( \zeta = \ldots + 1 + 1 + 1 + 1 + 1 + \ldots \) is a \( \zeta \)-sum of order types of rank smaller than 1, so its rank is also 1.
- \( \zeta + \zeta + \omega + 1 \) is a finite sum of order types of rank at most 1, so its rank is also 1.
- \( \zeta \times \omega \) is an \( \omega \)-sum of \( \zeta \)s, that is, \( \ldots + 0 + 0 + 0 + \zeta + \zeta + \ldots \), a \( \zeta \)-sum of summands having rank smaller than 2, so its rank is 2.
- \( \omega^n \) has rank \( n \).
- \( \omega^\omega = 1 + \omega + \omega^2 + \ldots \) has rank \( \omega \).

In general, if \( \alpha \) is an ordinal with Cantor normal form \( \alpha = \omega^{\alpha_1} \times n_1 + \ldots + \omega^{\alpha_k} \times n_k \), then its rank is \( \alpha_1 \).
Hausdorff’s theorem

Exactly the (countable) scattered order types have (a countable) rank.

Hence, when reasoning over scattered order types, one technique is to use induction on its rank. Well-founded induction can be used on well-ordered sets, like ordinals.

Why scattered order types?
Similarly to the technique that ordinals can be used to prove termination of “one-player” systems, scattered order types can be used to prove termination of some concurrent, “two-player” systems, where the aim of one player is to terminate the system, while the other tries to make it run indefinitely.
Some results on regular/context-free order types

**On regular order types**

- If $L_1$ and $L_2$ are regular languages, then it is decidable whether $o(L_1) = o(L_2)$ holds.
  
  Bloom–Choffrut, TCS, 2001

- An ordinal is regular if and only if it is smaller than $\omega^\omega$.
  
  Thomas, RAIRO, 1986

- Every scattered regular order type has rank smaller than $\omega$.
  
  but one cannot have all of them since there are uncountably many even for rank 2

- There is an operational characterization of scattered regular order types, involving $1$, $\omega$, $-\omega$, and the operations $+$ (binary sum), $\times \omega$ and $\times - \omega$.
  
  Heilbrunner, RAIRO, 1980
Some results on regular/context-free order types

On context-free order types

• It is undecidable for an input context-free language $L$ whether $o(L) = \eta$.
  Ésik, IPL, 2011

• It is decidable whether $o(L)$ is scattered, or well-ordered.
  Bloom–Ésik, Fundamenta Informaticæ, 2010; Bloom–Ésik, FICS 2009

• The rank of each deterministic context-free scattered language is smaller than $\omega^\omega$.
  Ésik, DLT, 2011; Bloom–Ésik, IJFCS, 2011

• The deterministic context-free ordinals are exactly those smaller than $\omega^{\omega^\omega}$.
  Ésik, DLT, 2011

• To each ordinal $o$ smaller than $\omega^{\omega^\omega}$ there exists a so-called “ordinal grammar” $G$ (whose nonterminals each generate a prefix-free language) with $o(G) = o$. But in general, there is no algorithm for transforming a context-free grammar generating a well-ordered language to an order equivalent ordinal grammar.
  Bloom–Ésik, Fundamenta Informaticæ, 2009
Main contributions

- The Hausdorff-rank of context-free ordinals is less than $\omega^\omega$.
  
  Ésik–Iván, LATIN 2012

  Thus, exactly the ordinals smaller than $\omega^\omega$ are the context-free ones.

- If $G$ is an ordinal grammar, then the Cantor normal form of $o(G')$ is effectively computable. Hence, the isomorphism problem of context-free ordinals is decidable if the ordinals are given by ordinal grammars.

  Gelle–Iván, TCS, 2019

- It is decidable whether $o(L)$ is a scattered context-free language of rank at most 1, and if so, then $o(L)$ is effectively computable as a finite sum of summands, each being $\omega$, $-\omega$ and 1.

  Gelle–Iván, GandALF 2019 and Gelle–Iván, SOFSEM 2020

- The rank of a scattered one-counter language is always smaller than $\omega^2$.

  Gelle–Iván, manuscript, submitted
Open questions

Open questions, short-term
- Characterize the scattered context-free order types of rank 2. Is their isomorphism problem decidable?
- Is there a way to compute $o(K \cup L)$ and $o(KL)$ effectively if both $K$ and $L$ are scattered context-free languages of known order types?

Open questions, long-term
- Is the isomorphism problem of scattered order types decidable?
- Is there an operational characterization of scattered context-free order types?
Some proof techniques

Techniques against scattered context-free languages

- If $G$ generates a scattered language, then for each rule $A \rightarrow \alpha$ there can be at most one nonterminal $B$ in $\alpha$ within the same component as $A$ (that is, with $B \Rightarrow^* uA\upsilon$ for some $u, \upsilon$).

- If $G$ generates a scattered language, then for each nonterminal $X$ there exists a word $u_X$ such that whenever $X \Rightarrow^* uX\beta$, then $u \in u_X^*$.

- For this definition of the rank we use, we have
  - $o(KL) = o(L) \times o(K)$ if $K$ is prefix-free (!)
  - the rank of $o(K \cup L)$ is at most the max rank of $o(K)$ and $o(L)$

- If $L^*$ is an infinite scattered language, then $L^* \subseteq u^*$ for some word $u$, hence it is a prefix chain and so $o(L^*) = \omega$

Most of our results went through by applying induction in a bottom-up way to the strongly connected components of the graph of $G$, and for each sentential form $\alpha$ at that level, reasoning about the possible order type of the language generated by $\alpha$. 
Thank you for your attention.