TREE AUTOMATA AND TERM REWRITE SYSTEMS

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I. TERM REWRITE SYSTEMS EFFECTIVELY PRESERVING RECOGNIZABILITY

Tree automata and recognizable tree languages proved to be an efficient tool in the theory of term rewrite systems. Let Σ be a ranked alphabet, let R be a term rewrite system over Σ, and let L be a tree language over Σ. Then $R^*_\Sigma(L)$ is the set of descendants of trees in L. A term rewrite system R over Σ preserves Σ-recognizability, if for each recognizable tree language L over Σ, $R^*_\Sigma(L)$ is recognizable. The signature $sign(R)$ of a term rewrite system R is the ranked alphabet consisting of all symbols appearing in the rules of R.

Let R be a term rewrite system over a ranked alphabet Σ. We say that R effectively preserves Σ-recognizability if for a given tree automaton $\mathcal{B}$ over Σ, we can effectively construct a tree automaton $\mathcal{C}$ over Σ such that $L(\mathcal{C}) = R^*_\Sigma(L(\mathcal{B}))$. Let R be a term rewrite system over a ranked alphabet Δ. We say that R effectively preserves recognizability if for a given ranked alphabet Σ with $sign(R) \subseteq \Sigma$ and a given tree automaton $\mathcal{B}$ over Σ, we can effectively construct a tree automaton $\mathcal{C}$ over Σ such that $L(\mathcal{C}) = R^*_\Sigma(L(\mathcal{B}))$. Let R be a term rewrite system over $sign(R)$, and let $\Sigma = \{ f, \sharp \} \cup sign(R)$, where $f \in \Sigma_2 - sign(R)$ and $\sharp \in \Sigma_0 - sign(R)$. We [3] showed that R effectively preserves Σ-recognizability if and only if R effectively preserves recognizability.

A term rewrite system R is gsm (generalized semi-monadic) if there is no rule $l \to r$ in R with $l \in X$ and the following holds. For any rules $l_1 \to r_1$ and $l_2 \to r_2$ in R, for any occurrences $\alpha \in O(r_1)$ and $\beta \in O(l_2)$, and for any supertree $l_3 \in T_\Sigma(X)$ of l_2/β with $var(l_3) \cap var(l_1) = \emptyset$, if

(i) $\alpha = \lambda$ or $\beta = \lambda$,

(ii) r_1/α and l_3 are unifiable, and

(iii) σ is a most general unifier of r_1/α and l_3,

then

(a) $l_2/\beta \in X$ or

(b) for each $\gamma \in O(l_3)$, if $l_2/\beta\gamma \in X$, then $\sigma(l_3/\gamma) \in X \cup T_\Sigma$.

We [3] showed that a linear gsm (lgsm) term rewrite system R over Δ effectively preserves recognizability in the following way. Let L be a recognizable tree language over Σ with $\Delta \subseteq \Sigma$, and let $\mathcal{B} = (\Sigma, B, R_\mathcal{B}, B')$ be a tree automaton recognizing L. Similarly to the constructions of Salomaa and Coquidé et al., we constructed a sequence of bottom-up tree automata $\mathcal{C}_i = (\Sigma, C, R_i, B')$, $i \geq 0$ having the same ranked alphabet, state set, and final state set. The rule set R_0 contains $R_\mathcal{B}$. Moreover, R_0 contains rules which enable R_0 to recognize the right-hand sides of rules in R. For each $i \geq 0$, R_{i+1} contains R_i, and for each rule $l \to r$ in R, $\mathcal{C}_{i+1}$ simulates, on the right-hand side r, the computation of $\mathcal{C}_i$ on the left-hand side l. There is a least integer $M \geq 0$ such that $R_M = R_{M+1}$. Hence $\mathcal{C}_M = \mathcal{C}_{M+1}$. We [3] showed that $L(\mathcal{C}_M) = R^*_\Sigma(L)$.

Brainerd, Kozen, and Fülöp and Vágvölgyi showed that a tree language L is recognizable if and only if there exists a ground term rewrite system R such that L is the union of finitely many $\leftrightarrow^*_R$-classes. We [3] showed that a tree language L is recognizable if and only if there exists a term rewrite system R such that $R \cup R^{-1}$ is an lgsm term rewrite system and that L is the union of finitely many $\leftrightarrow^*_R$-classes.

We [3] showed the following decidability results.

(1) Let R_1, R_2 be term rewrite systems. Let R_1 effectively preserve recognizability. Then it is decidable if $\to^*_{R_2} \subseteq \to^*_{R_1}$.

(2) Let R_1 and R_2 be term rewrite systems effectively preserving recognizability. Then it is decidable which one of the following four mutually excluding conditions hold.

(i) $\to^*_{R_1} \subset \to^*_{R_2}$,

(ii) $\to^*_{R_2} \subset \to^*_{R_1}$,

(iii) $\to^*_{R_1} = \to^*_{R_2}$,

(iv) $\to^*_{R_1} \bowtie \to^*_{R_2}$,

where "$\bowtie$" stands for the incomparability relationship.

(3) For an lgsm term rewrite system R, it is decidable whether R is left-to-right minimal. (A term rewrite system R is left-to-right minimal if for each rule $l \to r$ in R, $\to^*_{R-\{ l\to r \}} \subset \to^*_R$.)

(4) Let R_1 and R_2 be term rewrite systems such that $R_1 \cup R_1^{-1}$ and $R_2 \cup R_2^{-1}$ are term rewrite systems and effectively preserve recognizability. Then it is decidable which one of the following four mutually excluding conditions holds.

(i) $\leftrightarrow^*_{R_1} \subset \leftrightarrow^*_{R_2}$,

(ii) $\leftrightarrow^*_{R_2} \subset \leftrightarrow^*_{R_1}$,

(iii) $\leftrightarrow^*_{R_1} = \leftrightarrow^*_{R_2}$,

(iv) $\leftrightarrow^*_{R_1} \bowtie \leftrightarrow^*_{R_2}$.

(5) Let R be a term rewrite system such that $R \cup R^{-1}$ is an lgsm term rewrite system. Then it is decidable whether R is two-way minimal. (A term rewrite system R is two-way minimal if for each rule $l \to r$ in R, $\leftrightarrow^*_{R-\{ l\to r \}} \subset \leftrightarrow^*_R$.)

(6) Let R_1, R_2 be term rewrite systems over a ranked alphabet Σ. Let R_1 effectively preserve recognizability. Let $g \in \Sigma - \Sigma_0$ be such that g does not occur on the left-hand side of any rule in R_1, and let $\sharp \in \Sigma_0$ be irreducible for R_1. Then it is decidable if $\to^*_{R_2} \cap (T_\Sigma \times T_\Sigma) \subseteq \to^*_{R_1} \cap (T_\Sigma \times T_\Sigma)$.

(7) Let R_1 and R_2 be term rewrite systems over Σ effectively preserving recognizability. Moreover, let $g_1, g_2 \in \Sigma - \Sigma_0$ be such that for each $i \in \{ 1, 2 \}$, g_i does not occur on the left-hand side of any rule in R_i. Let $\sharp_1, \sharp_2 \in \Sigma_0$ be such that for each $i \in \{ 1, 2 \}$, $\sharp_i$ is irreducible for R_i. Then it is decidable which one of the following four mutually excluding conditions hold.

(i) $\to^*_{R_1} \cap (T_\Sigma \times T_\Sigma) \subset \to^*_{R_2} \cap (T_\Sigma \times T_\Sigma)$,

(ii) $\to^*_{R_2} \cap (T_\Sigma \times T_\Sigma) \subset \to^*_{R_1} \cap (T_\Sigma \times T_\Sigma)$,

(iii) $\to^*_{R_1} \cap (T_\Sigma \times T_\Sigma) = \to^*_{R_2} \cap (T_\Sigma \times T_\Sigma)$,

(iv) $\to^*_{R_1} \cap (T_\Sigma \times T_\Sigma) \bowtie \to^*_{R_2} \cap (T_\Sigma \times T_\Sigma)$.

(8) Let R be an lgsm term rewrite system over Σ. Moreover, let $g \in \Sigma - \Sigma_0$ be such that g does not occur on the left-hand side of any rule in R, and let $\sharp \in \Sigma_0$ be irreducible for R. Then it is decidable whether R is left-to-right ground minimal. (A term rewrite system R over Σ is left-to-right ground minimal if for each rule $l \to r$ in R, $\to^*_{R-\{ l\to r \}} \cap (T_\Sigma \times T_\Sigma) \subset \to^*_R \cap (T_\Sigma \times T_\Sigma)$.)

(9) Let R be a term rewrite system over Σ such that $R \cup R^{-1}$ is an lgsm term rewrite system. Moreover, let

$g \in \Sigma - \Sigma_0$ be such that g does not occur in any rule of R, and let $\sharp \in \Sigma_0$ be irreducible for $R \cup R^{-1}$. Then it is decidable whether R is two-way ground minimal. (A term rewrite system R over Σ is two-way ground minimal if for each rule $l \to r$ in R, $\leftrightarrow^*_{R-\{\,l\to r\,\}} \cap (T_\Sigma \times T_\Sigma) \subset \leftrightarrow^*_R \cap (T_\Sigma \times T_\Sigma)$.)

(10) Let R be a term rewrite system over Σ effectively preserving recognizability, and let $p, q \in T_\Sigma(X)$. Then it is decidable if there exists a tree $r \in T_\Sigma(X)$ such that $p \to^*_R r$ and $q \to^*_R r$.

(11) Let R be a term rewrite system over Σ effectively preserving recognizability. Then it is decidable if R is locally confluent.

Let R be a term rewrite system over $sign(R)$, and let $\Sigma = \{\,f, \sharp\,\} \cup sign(R)$, where $f \in \Sigma_2 - sign(R)$ and $\sharp \in \Sigma_0 - sign(R)$. We [3] showed that R effectively preserves Σ-recognizability if and only if R effectively preserves recognizability. We [4] improved this result for left-linear term rewrite systems. We showed the following. Let R be a left-linear term rewrite system over $sign(R)$, and let $\Sigma = \{\,g, \sharp\,\} \cup sign(R)$, where $g \in \Sigma_1 - sign(R)$ and $\sharp \in \Sigma_0 - sign(R)$. Then R effectively preserves Σ-recognizability if and only if R effectively preserves recognizability. This result makes it easier to show that a given left-linear term rewrite system effectively preserves recognizability. Furthermore, R preserves Σ-recognizability if and only if R preserves recognizability.

We [6] introduced and studied the half-monadic term rewrite system, which is an extension of the right-ground term rewrite system and is a slight extension of the semi-monadic term rewrite system. Takai, Kaji, and Seki introduced the concept of the finite path overlapping term rewrite system, and showed that right-linear finite path overlapping term rewrite systems effectively preserve recognizability. A half-monadic term rewrite system is a finite path overlapping term rewrite system as well. Hence right-linear half-monadic term rewrite systems effectively preserve recognizability. We [6] showed that termination and convergence are decidable properties for right-linear half-monadic term rewrite systems.

II. GROUND TERM REWRITE SYSTEMS

In paper [5], we showed that a reduced ground term rewrite system over some ranked alphabet Σ has a good property: if we omit any rules, the omitted rules and the remaining rules generate congruence relations such that their intersection is the identity relation on T_Σ. In general, given two ground term rewrite systems A and B over some ranked alphabet Σ with $\leftrightarrow^*_A \subseteq \leftrightarrow^*_B$, one may want to eliminate all nontrivial pairs of $\leftrightarrow^*_A$ from $\leftrightarrow^*_B$. In other words, one may want to construct a ground term rewrite system (gtrs for short) C over Σ such that $\leftrightarrow^*_{A\cup C} = \leftrightarrow^*_B$ and that $\leftrightarrow^*_A \cap \leftrightarrow^*_C$ is the identity relation on T_Σ. Let A and B be gtrs's over some ranked alphabet Σ with $\leftrightarrow^*_A \subseteq \leftrightarrow^*_B$. We say that a gtrs C over Σ is a congruential complement of A for B, if $\leftrightarrow^*_{A\cup C} = \leftrightarrow^*_B$ and $\leftrightarrow^*_A \cap \leftrightarrow^*_C$ is the identity relation over T_Σ. We note that A may have more than one congruential complement for B, which are pairwise nonequivalent. Consider the following example. Let $\Sigma = \{\,a, b, f\,\}$, where a, b are of rank 0, and f is of rank 1. Consider the gtrs's

$$A = \{\, f(a) \to a,\ \ f(b) \to b \,\}\ ,$$

$$B = \{\, f(a) \to a,\ \ f(b) \to b,\ \ a \to b \,\}\ ,$$

$$C = \{\, a \to b \,\}\ ,$$

$$D = \{\, a \to f(b) \,\}\ ,$$

$$E = \{\, f(a) \to f^3(b) \,\}$$

over Σ. Then each one of C, D, and E is a congruential complement of A for B. Moreover, $\leftrightarrow^*_C \neq \leftrightarrow^*_D$, $\leftrightarrow^*_C \neq \leftrightarrow^*_E$, and $\leftrightarrow^*_D \neq \leftrightarrow^*_E$.

Our [5] main result is the following. Given gtrs's A, B, C over some ranked alphabet Σ with $\leftrightarrow^*_A \subseteq \leftrightarrow^*_B$, one can effectively decide if C is a congruential complement of A for B.

In paper [1] we studied that version of the ground tree transducer game where the same tree automaton appears as the first and second component of the associated ground tree transducer. We gave conditions which imply that Beta has a winning strategy. Furthermore, we showed the following decidability result. Given a ground tree transducer game where the underlying tree automaton A cannot evaluate some tree into a state or A is deterministic, we can decide which player has a winning strategy. Moreover, whatever player has a winning strategy, we can effectively construct a partial recursive winning strategy for him.

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