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ALGEBRAIC AND GRAPH-THEORETIC PROPERTIES OF INFINITE n-POSETS *

Zoltán Ésik[1] and Zoltán L. Németh[1]

Abstract. A Σ-labeled n-poset is an (at most) countable set, labeled in the set Σ, equipped with n partial orders. The collection of all Σ-labeled n-posets is naturally equipped with n binary product operations and n ω-ary product operations. Moreover, the ω-ary product operations give rise to n ω-power operations. We show that those Σ-labeled n-posets that can be generated from the singletons by the binary and ω-ary product operations form the free algebra on Σ in a variety axiomatizable by an infinite collection of simple equations. When $n = 1$, this variety coincides with the class of ω-semigroups of Perrin and Pin. Moreover, we show that those Σ-labeled n-posets that can be generated from the singletons by the binary product operations and the ω-power operations form the free algebra on Σ in a related variety that generalizes Wilke's algebras. We also give graph-theoretic characterizations of those n-posets contained in the above free algebras. Our results serve as a preliminary study to a development of a theory of higher dimensional automata and languages on infinitary associative structures.

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Introduction

The input structure of an automaton is usually a free algebra, freely generated by some finite alphabet Σ in a variety of algebras equipped with a finite number of operations. For example, in classical automata theory, the input structures can be identified as the finitely generated free semigroups, or monoids. Regarding

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[1] Department of Computer Science, University of Szeged, P.O.B. 652, 6701 Szeged, Hungary; `zlnemeth@inf.u-szeged.hu`

automata on ω-words, Perrin and Pin [15, 16] proposed to use free ω-semigroups, which are a two-sorted generalization of semigroups and possess both a binary and an ω-ary product operation. Independently, Wilke proposed to use another kind of two-sorted algebras, see [18], freely generated in an infinitely axiomatizable variety equipped with only finitary operations, a binary product and a unary ω-power operation. It has been shown by Wilke that the category of finite Wilke algebras is equivalent to the category of finite ω-semigroups of Perrin and Pin. Here, we propose an extension of the algebras of Perrin, Pin and Wilke as the basic underlying variety for a generalization of the classical framework to higher dimensional automata on infinitary associative structures. This may be seen as a continuation of the paper [3], where we treated higher dimensional automata on finitary associative input structures. Our work is closely related to [7, 8].

We call a set equipped with n associative operations, among which there is no further connection, an n-semigroup. A concrete description of freely generated n-semigroups, based on the notion of labeled n-posets, was given in [2]. Here, a Σ-labeled n-poset, where Σ is a set of labels, consists of a set P equipped with n partial orders and a labeling function $P \to A$. We usually identify isomorphic Σ-labeled n-posets. We define two types of operations on n-posets: n binary product operations, and n ω-power operations, where a product and an ω-power operation corresponds to each partial order. It was shown in [2] that a subcollection of finite Σ-labeled n-posets, equipped with the binary product operations, forms the Σ-generated free n-semigroup. Here, we show that those finite or infinite Σ-labeled n-posets that can be generated from the singletons by the binary and ω-ary product operations form the free algebra on Σ in the variety of n-ω-semigroups axiomatizable by an infinite collection of simple equations. When $n = 1$, this variety coincides with the class of ω-semigroups of Perrin and Pin. Moreover, we show that those Σ-labeled n-posets that can be generated from the singletons by the binary product operations and the ω-power operations form the free algebra on Σ in a related variety that generalizes Wilke's algebras. We also give graph-theoretic characterizations of those n-posets contained in the above free algebras. These characterizations are related to the characterization of series-parallel posets as the finite N-free posets, *cf.* [5, 6, 17], and to the characterizations obtained in [1, 4]. Our results serve as a preliminary study to a development of a theory of higher dimensional automata and languages on infinitary structures.

Automata on series-parallel posets were studied by K. Lodaya and P. Weil in [12–14]. Their work was extended into two directions by D. Kuske [9–11], to automata on infinite posets and to (first- and second-order) logical definability.

1. n-POSETS

Some notation. In the sequel, n always denotes a positive integer and Σ a finite alphabet. We write $[n]$ for the set $\{1, 2, \ldots, n\}$.

Let Σ denote a nonempty set of labels. A Σ*-labeled* n*-poset*, or n-poset, for short, is an (at most) countable set P of vertices equipped with n (irreflexive) partial orders $<_i^P$, $i \in [n]$, and a labeling function $\lambda_P : P \to \Sigma$. We denote an n-poset variously by $(P, <_1^P, \ldots, <_n^P, \lambda_P)$ or $(P, <_1, \ldots, <_n, \lambda)$, or by just $(P, <_1, \ldots, <_n)$ or P. We say that P is a *complete* n-poset if for every $u, v \in P$ with $u \neq v$ there is exactly one $i \in [n]$ such that $u <_i v$ or $v <_i u$ holds.

A *morphism* between n-posets is a function on the vertices that preserves the partial orders and the labeling. An *isomorphism* is a bijective morphism whose inverse is also a morphism. Below we will usually identify isomorphic n-posets. We say that P is an *(induced) sub-n-poset of* Q if $P \subseteq Q$ and the partial orders $<_i^P$ are the restrictions of the corresponding orders $<_i^Q$, moreover, λ_P is the restriction of λ_Q to the set P.

For each $i \in [n]$, we define a *binary* $\cdot_i$*-product*, an ω*-ary* $\cdot_i$*-product*, and a *unary* ω_i*-power* operation on the collection of all n-posets.

$\cdot_i$-PRODUCT. Suppose that $P = (P, <_1^P, \ldots, <_n^P, \lambda_P)$ and $Q = (Q, <_1^Q, \ldots, <_n^Q, \lambda_Q)$ are n-posets. Without loss of generality, assume that P and Q are disjoint. We define the $\cdot_i$*-product* $P \cdot_i Q$ to be the n-poset with underlying set $P \cup Q$, partial orders

$$<_j^{P \cdot_i Q} = \begin{cases} <_j^P \cup <_j^Q & \text{if } j \neq i \\ <_i^P \cup <_i^Q \cup (P \times Q) & \text{if } j = i, \end{cases} \qquad j \in [n],$$

and labeling $\lambda_{P \cdot_i Q} = \lambda_P \cup \lambda_Q$.

It is clear that each $\cdot_i$-product operation is associative.

ω-ARY $\cdot_i$-PRODUCT. Suppose that $P_1, P_2, \ldots$ are n-posets. We may assume that they are pairwise disjoint. For each $i \in [n]$, the ω*-ary* $\cdot_i$*-product* is defined by

$$\omega_i(P_1, P_2, \ldots) = (P_1 \cup P_2 \cup \ldots, <'_1, \ldots, <'_n, \lambda'),$$

where for any $x \in P_k$ and $y \in P_l$ $(k, l \geq 1)$,

$$x <'_j y \Leftrightarrow (k = l \text{ and } x <_j y \text{ in } P_k) \text{ or } (j = i \text{ and } k < l), \quad j \in [n],$$

and

$$\lambda' = \lambda_{P_1} \cup \lambda_{P_2} \cup \ldots$$

In order to simplify the treatment, we will also use the notation $P_1 \cdot_i P_2 \cdot_i \ldots$ for $\omega_i(P_1, P_2, \ldots)$.

ω_i-POWER. If P is any n-poset, then the ω_i*-power* of P is defined as the ω-ary $\cdot_i$-product of ω copies of P:

$$P^{\omega_i} = P \cdot_i P \cdot_i \ldots$$

Let $\mathbf{Pos}_{n,F}(\Sigma)$ and $\mathbf{Pos}_{n,I}(\Sigma)$ denote the sets of all nonempty, finite and countably infinite Σ-labeled n-posets, respectively.

2. Constructible n-posets

Below we will restrict the operations $P \cdot_i Q$ and P^{ω_i} to finite n-posets P. The n-poset Q may be either finite or infinite. Similarly, we allow the formation of an ω-ary product $P_1 \cdot_i P_2 \cdot_i \ldots$ only when each P_i is finite.

We call a Σ-labeled n-poset *constructible* if it can be generated from the singleton Σ-labeled n-posets by the binary and ω-ary $\cdot_i$-product operations. Moreover, we call a Σ-labeled n-poset *strictly constructible* if it can be generated from the singletons by the binary $\cdot_i$-product operations and the ω_i-power operations. It is obvious that every strictly constructible n-poset is constructible. Moreover, a finite n-poset is constructible iff it is strictly constructible iff it can be generated from the singletons by the binary $\cdot_i$-product operations only. Note that a constructible n-poset is nonempty.

Example 2.1. Let $P_n = a \cdot_1 (b \cdot_2 \ldots \cdot_2 b) \cdot_1 a$, for each $n \geq 1$, where a and b denote the singleton 2-posets labeled a and b, respectively, and where there are n copies of b. Then each P_n is constructible. Moreover, $P_1 \cdot_1 P_2 \cdot_1 \ldots$ is constructible and $P_1 \cdot_1 P_2^{\omega_2}$ is strictly constructible. Note that $a^{\omega_1} \cdot b$ is not constructible since a^{ω_1} is infinite.

We define $\mathbf{SP}_{n,F}(\Sigma)$ to be the set of all finite constructible Σ-labeled n-posets, and denote by $\omega\mathbf{SP}_{n,I}(\Sigma)$ the set of all infinite constructible Σ-labeled n-posets. Similarly, we let $\mathbf{SP}^{\omega}_{n,I}(\Sigma)$ denote the set of all infinite strictly constructible Σ-labeled n-posets.

For any finite constructible n-poset P, let $\mathrm{rank}(P)$ denote the *rank* of P, *i.e.*, the least number of binary product operations needed to construct P from the singletons. When P is an infinite constructible n-poset, say that P is *primitive* if it is of the form

$$P_1 \cdot_i P_2 \cdot_i \ldots$$

for some $i \in [n]$ and finite constructible n-posets $P_1, P_2, \ldots$ Now each infinite constructible n-poset can be generated from the primitive infinite constructible n-posets by multiplication with finite constructible n-posets on the left. We define the rank of an infinite constructible n-poset P as the least number of left multiplications with finite constructible n-posets needed to construct P from the primitive infinite n-posets.

Lemma 2.2. *Every constructible n-poset is complete.*

Proof. By induction on the rank. □

Lemma 2.3. *Suppose that* $(P, <_1, \ldots, <_n)$ *is constructible. Then the relation* $< = <_1 \cup \ldots \cup <_n$ *is a total order relation on* P.

Proof. By induction on the rank. □

When the total order $<$ has a greatest element v, then we say that v is the greatest element of P.

Suppose that P is a Δ-labeled n-poset and for each vertex v in P, Q_v is a Σ-labeled n-poset. Then we denote by $P[Q_v/v]_{v\in P}$ the Σ-labeled n-poset obtained from P by replacing each vertex v in P by Q_v. The following fact is clear.

Lemma 2.4. *Suppose that $P_1, P_2, \ldots$ are (disjoint) n-posets and for each vertex v in P_i, $i = 1, 2, \ldots$, Q_v is an n-poset. Let $R = P_1 \cdot_i P_2$ and $R' = P_1 \cdot_i P_2 \cdot_i \ldots$, where $i \in [n]$. Then*

$$\begin{aligned} R[Q_v/v]_{v\in R} &= P_1[Q_v/v]_{v\in P_1} \cdot_i P_2[Q_v/v]_{v\in P_2} \\ R'[Q_v/v]_{v\in R'} &= P_1[Q_v/v]_{v\in P_1} \cdot_i P_2[Q_v/v]_{v\in P_2} \cdot_i \ldots \end{aligned}$$

Lemma 2.5. *Suppose that P is a constructible n-poset and for each vertex v in P, Q_v is a constructible n-poset. Suppose that Q_v is finite whenever v is not the greatest vertex of P. Then $P[Q_v/v]_{v\in P}$ is constructible.*

Proof. First assume that P is finite. We argue by induction on $k = \mathrm{rank}(P)$. When $k = 0$, then P is a singleton and our claim is obvious. When $k > 0$, $P = P_1 \cdot_i P_2$, for some finite constructible n-posets P_1 and P_2 with $\mathrm{rank}(P_1), \mathrm{rank}(P_2) < k$. By Lemma 2.4 we have $P[Q_v/v] = P_1[Q_v/v]_{v\in P_1} \cdot_i P_2[Q_v/v]_{v\in P_2}$. Now, Q_v is finite for each v in P_1, and Q_v is finite for each v in P_2 such that v is not the greatest vertex of P_2. Thus, by induction, both $P_1[Q_v/v]_{v\in P_1}$ and $P_2[Q_v/v]_{v\in P_2}$ are constructible. Also, we have that $P_1[Q_v/v]_{v\in P_1}$ is finite. It follows that $P[Q_v/v]_{v\in P}$ is also constructible.

Assume now that P is infinite. We again argue by induction on $k = \mathrm{rank}(P)$. If $k = 0$ then $P = P_1 \cdot_i P_2 \cdot_i \ldots$, where each P_j is finite and constructible. Clearly, each Q_v, $v \in P$ is also finite, and each $P_j[Q_v/v]_{v\in P_j}$ is finite and constructible. It follows by Lemma 2.4 and the induction hypothesis that $P[Q_v/v]_{v\in P} = P_1[Q_v/v]_{v\in P_1} \cdot_i P_2[Q_v/v]_{v\in P_2} \cdot_i \ldots$ is constructible. Assume now that $k > 0$. Then $P = P_1 \cdot_i P_2$, for some finite constructible n-poset P_1 and infinite constructible n-poset P_2 with $\mathrm{rank}(P_2) < k$. Now $P_1[Q_v/v]_{v\in P_1}$ is finite and constructible by the first case, while $P_2[Q_v/v]_{v\in P_2}$ is infinite and constructible by the induction hypothesis. It follows that $P[Q_v/v]_{v\in P}$ is constructible. $\square$

3. Freeness

We start our discussion with the finite case. The notion of semigroup can naturally be generalized to the case of any number of associative operations. We define an *n-semigroup* to be an algebraic structure consisting of a set and n associative binary product operations, usually denoted $\cdot_i$, $i \in [n]$. A morphism of n-semigroups preserves all operations.

Below we identify each letter $a \in \Sigma$ with the singleton n-poset labeled a. We need the following result.

Theorem 3.1 [2]**.** *For each nonempty set Σ, $\mathbf{SP}_{n,F}(\Sigma)$, equipped with the operations $\cdot_i$ defined above, is freely generated by Σ in the variety of all n-semigroups.*

Recall that $\mathbf{Pos}_{n,F}(\Sigma)$ and $\mathbf{Pos}_{n,I}(\Sigma)$ denote the set of all finite and countably infinite Σ-labeled n-posets, respectively. Equipped with the binary and the ω-ary $\cdot_i$-product operations, for all $i \in [n]$, these sets form a two-sorted algebra

$$\omega\mathbf{Pos}_n(\Sigma) = (\mathbf{Pos}_{n,F}(\Sigma), \mathbf{Pos}_{n,I}(\Sigma), \cdot_1, \ldots, \cdot_n, \omega_1, \ldots, \omega_n),$$

where the $\cdot_i$-product operations are appropriately polymorphic. (We use the same notation for the product of two finite n-posets and for the product of a finite and an infinite n-poset.) It is easily seen that the algebra $\omega\mathbf{Pos}_n(\Sigma)$ satisfies the following equations:

$$x \cdot_i (y \cdot_i u) = (x \cdot_i y) \cdot_i u, \tag{1}$$

$$x \cdot_i \omega_i(x_1, x_2, \ldots) = \omega_i(x, x_1, x_2, \ldots), \tag{2}$$

$$\begin{aligned}\omega_i(x_1 \cdot_i \ldots \cdot_i x_{k_1-1},\, x_{k_1} \cdot_i \ldots \cdot_i x_{k_2-1},\, \ldots) = \omega_i(&x_1, \ldots, x_{k_1-1},\, x_{k_1},\\ &x_{k_1+1}, \ldots, x_{k_2-1},\, \ldots),\end{aligned} \tag{3}$$

for all $x, y, x_1, x_2, \ldots \in \mathbf{Pos}_{n,F}(\Sigma)$, $u \in \mathbf{Pos}_{n,F}(\Sigma) \cup \mathbf{Pos}_{n,I}(\Sigma)$ and $i \in [n]$, and for all increasing sequences of positive integers $k_1 < k_2 < \ldots$ This motivates the following definition.

Definition 3.2. Call an algebra $\mathcal{C} = (C_F, C_I, \cdot_1, \ldots, \cdot_n, \omega_1, \ldots, \omega_n)$ an *n-ω-semigroup* if it satisfies equations (1)–(3) above, where $x, y, x_1, x_2, \ldots$ range over C_F and both $u \in C_F$ and $u \in C_I$ are allowed. A morphism of n-ω-semigroups $\mathcal{C} \to \mathcal{D}$, where $\mathcal{D} = (D_F, D_I, \cdot_1, \ldots, \cdot_n, \omega_1, \ldots, \omega_n)$, is a pair of functions $h = (h_F : C_F \to D_F, h_I : C_I \to D_I)$ that jointly preserve the operations.

Remark 3.3. When $n = 1$, an n-ω-semigroup is just an ω-semigroup as defined in [16].

Note that the smallest subalgebra of $\omega\mathbf{Pos}_n(\Sigma)$ containing the singleton n-posets labeled with the elements of Σ is the algebra $\omega\mathbf{SP}_n(\Sigma) = (\mathbf{SP}_{n,F}(\Sigma), \omega\mathbf{SP}_{n,I}(\Sigma), \cdot_1, \ldots, \cdot_n, \omega_1, \ldots, \omega_n)$ of all constructible Σ-labeled n-posets.

Suppose that P is an n-poset. We call P *$\cdot_i$-irreducible* if P is nonempty and has no decomposition $P_1 \cdot_i P_2$, where P_1 and P_2 are nonempty. Note that for every $i \in [n]$, each finite constructible n-poset has a unique, possibly trivial decomposition into the $\cdot_i$-product of $\cdot_i$-irreducible n-posets.

We omit the proof of the following lemma.

Lemma 3.4. *Any $P \in \mathbf{SP}_{n,I}(\Sigma)$ can uniquely be written as*

$$P = P_1 \cdot_i P_2 \cdot_i \ldots, \tag{4}$$

where $i \in [n]$ and each $P_k \in \mathbf{SP}_{n,F}(\Sigma)$ is $\cdot_i$-irreducible, or as

$$P = P_1 \cdot_i P_2 \cdot_i \ldots \cdot_i P_r \cdot_i Q, \tag{5}$$

where $i \in [n]$, $r > 0$, moreover, $P_k \in \mathbf{SP}_{n,F}(\Sigma)$, $k \in [r]$ and $Q \in \mathbf{SP}_{n,I}(\Sigma)$ are $\cdot_i$-irreducible. Moreover, in case (5), the rank of Q is strictly less than the rank of P.

Theorem 3.5. *The algebra $\omega\mathbf{SP}_n(\Sigma)$ if freely generated by Σ in the variety of all n-ω-semigroups.*

Proof. Suppose that $\mathbf{C} = (C_F, C_I)$ is an n-ω-semigroup. We show how to extend any function $h : \Sigma \to C_F$ to a homomorphism $h^\sharp = (h^\sharp_F, h^\sharp_I) : \omega\mathbf{SP}(\Sigma) \to \mathbf{C}$. Indeed, by Theorem 3.1, there exists a unique n-semigroup homomorphism $h^\sharp_F$: $\mathbf{SP}_{n,F}(\Sigma) \to C_F$ which extends h. We need to define $h^\sharp_I : \mathbf{SP}_{n,I}(\Sigma) \to C_I$. The definition is based on Lemma 3.4. In case (4), let

$$h^\sharp_I(P) = h^\sharp_F(P_1) \cdot_i h^\sharp_F(P_2) \cdot_i \dots$$

In case (5), let

$$h^\sharp_I(P) = h^\sharp_F(P_1) \cdot_i h^\sharp_F(P_2) \cdot_i \dots \cdot_i h^\sharp_F(P_r) \cdot_i h^\sharp_I(Q),$$

where, using induction on the rank of P, we can assume that $h^\sharp_I(Q)$ is already defined.

It is easy to verify that $h^\sharp$ is indeed a homomorphism. Here, we only demonstrate that $h^\sharp$ preserves the ω-ary product operations, *i.e.*,

$$h^\sharp_I(P_1 \cdot_i P_2 \cdot_i \dots) = h^\sharp_F(P_1) \cdot_i h^\sharp_F(P_2) \cdot_i \dots, \tag{6}$$

for all $i \in [n]$ and finite constructible n-posets $P_1, P_2, \dots$ Since each P_k, $k \geq 1$ is finite, let $P_k = P_{k,1} \cdot_i P_{k,2} \cdot_i \dots \cdot_i P_{k,s_k}$, $s_k \geq 1$, be the unique decomposition of P_k into a $\cdot_i$-product of $\cdot_i$-irreducible n-posets. By definition, the left-hand side of (6) is equal to

$$h^\sharp_F(P_{1,1}) \cdot_i \dots \cdot_i h^\sharp_F(P_{1,s_1}) \cdot_i h^\sharp_F(P_{2,1}) \cdot_i \dots \cdot_i h^\sharp_F(P_{2,s_2}) \cdot_i \dots$$

Moreover, using the fact that $h^\sharp_F$ is a homomorphism of n-semigroups, the right-hand side of (6) is equal to

$$(h^\sharp_F(P_{1,1}) \cdot_i \dots \cdot_i h^\sharp_F(P_{1,s_1})) \cdot_i (h^\sharp_F(P_{2,1}) \cdot_i \dots \cdot_i h^\sharp_F(P_{2,s_2})) \cdot_i \dots$$

Thus, the two sides are equal by axiom (3). With the help of (1) and (2), one can prove similarly that $h^\sharp_I(P \cdot_i Q) = h^\sharp_F(P) \cdot_i h^\sharp_I(Q)$ for all constructible P, Q such that P is finite and Q is infinite. $\square$

One can consider the ω_i-power operations instead of the ω-$\cdot_i$-products. Let

$$\mathbf{Pos}^\omega_n(\Sigma) = (\mathbf{Pos}_{n,F}(\Sigma), \mathbf{Pos}_{n,I}(\Sigma), \cdot_1, \dots, \cdot_n, {}^{\omega_1}, \dots, {}^{\omega_n}).$$

Now $\mathbf{Pos}_n^\omega(\Sigma)$ satisfies (1) and the following equations:

$$(x \cdot_i y)^{\omega_i} = x \cdot_i (y \cdot_i x)^{\omega_i}, \tag{7}$$

$$(\underbrace{x \cdot_i x \cdot_i \ldots \cdot_i x}_{k \text{ times}})^{\omega_i} = x^{\omega_i}, \quad k \geq 2, \tag{8}$$

for all $x, y \in \mathbf{Pos}_{n,F}(\Sigma)$ and $i \in [n]$.

Definition 3.6. Call an algebra $\mathcal{C} = (C_F, C_I, \cdot_1, \ldots, \cdot_n, {}^{\omega_1}, \ldots, {}^{\omega_n})$ an *n-Wilke-algebra* if it satisfies equations (1), (7) and (8) above. A *morphism* of n-Wilke-algebras preserves all operations.

Note that every n-ω-semigroup can be viewed as an n-Wilke algebra, where the ω-power operations are defined naturally with the help of the ω-product operations: $x^{\omega_i} = x \cdot_i x \cdot_i \ldots$ Then the equations (2) and (3) imply equations (7) and (8). In fact, the argument from [18] establishing the equivalence of the category of finite ω-semigroups and the category of finite Wilke-algebras is easily applicable to show that this result generalizes to n-ω-semigroups and n-Wilke-algebras.

Proposition 3.7. *For each n, the category of finite n-ω-semigroups and the category of finite n-Wilke-algebras are equivalent.*

The smallest subalgebra of $\mathbf{Pos}_n^\omega(\Sigma)$ containing the singletons labeled with the elements of Σ is the algebra $\mathbf{SP}_n^\omega(\Sigma) = (\mathbf{SP}_{n,F}(\Sigma), \mathbf{SP}_{n,I}^\omega(\Sigma), \cdot_1, \ldots, \cdot_n, {}^{\omega_1}, \ldots, {}^{\omega_n})$ of strictly constructible n-posets.

Let us now define the (strict) rank of an infinite strictly constructible n-poset P as the least number of binary product and ω-power operations needed to generate P from the singletons.

Lemma 3.8. *Any $P \in \mathbf{SP}_{n,I}^\omega(\Sigma)$ can be uniquely written either as*

$$P = P_1 \cdot_i \ldots \cdot_i P_k \cdot_i (P_{k+1} \cdot_i \ldots \cdot_i P_m)^{\omega_i},$$

where $i \in [n]$ and $P_1, \ldots, P_m \in \mathbf{SP}_{n,F}(\Sigma)$ are $\cdot_i$-irreducible, moreover, P_k is not isomorphic to P_m and there is no proper divisor d of m such that $(P_{k+1} \cdot_i \ldots \cdot_i P_{k+d})^{m/d}$ is isomorphic to $(P_{k+1} \cdot_i \ldots \cdot_i P_m)$, or as

$$P = P_1 \cdot_i P_2 \cdot_i \ldots \cdot_i P_r \cdot_i Q,$$

where $i \in [n]$, $r > 0$, moreover, $P_k \in \mathbf{SP}_{n,F}(\Sigma)$, $k \in [r]$ and $Q \in \mathbf{SP}_{n,I}^\omega(\Sigma)$ are $\cdot_i$-irreducible, and the rank of Q is strictly less than the rank of P.

Using the above lemma, the following theorem can be proved similarly to the proof of Theorem 3.5.

Theorem 3.9. *The algebra $\mathbf{SP}_n^\omega(\Sigma)$ if freely generated by Σ in the variety of n-Wilke-algebras.*

4. A characterization

In this section, we recall the graph-theoretic characterization of the finite constructible n-posets from [2] and provide a related characterization of the infinite constructible and strictly constructible n-posets.

Note that a (strict, countable) partially ordered set, or a poset, is just an n-poset for $n = 1$, without the labeling function, or with a singleton set of labels and a trivial labeling function.

A *filter* of a poset is a nonempty upward closed subset, while an *ideal* is a nonempty downward closed subset. Each nonempty subset of a poset is included in a smallest filter and a smallest ideal, respectively called the filter and the ideal *generated* by the set. A filter is called a *principal filter* if it is generated by a single element, or more precisely, by a singleton set. *Principal ideals* are defined in the same way. We say that a nonempty set $X \subseteq P$ is *directed* if any two elements of X have an upper bound in X. A poset $(P, <)$ is *connected* if the symmetric closure of the order relation $<$ defines a connected graph on P. The *connected components* of a poset P are the maximal connected subposets of P. A *chain* is a totally ordered poset, and an ω*-chain* is a chain isomorphic to the usual order of the naturals. An *antichain* is a poset such that no two elements are related by the order relation.

We say that a poset $(P, <)$ *has a linearization to an* ω*-chain* if there is an extension of the partial order $<$ to an order $<'$ such that $(P, <')$ is an ω-chain.

Lemma 4.1. *A poset P has a linearization to an ω-chain iff it is countably infinite and each principal ideal of P is finite.*

A poset $(P, <)$ satisfies the *N-condition*, or is *N-free*, if it has no "N's", *i.e.*, there is no four element subset $\{a, b, c, d\}$ of P whose only order relations are $a < c$, $b < c$ and $b < d$. We say that an n-poset P satisfies the *triangle condition* if any three different vertices u, v and w of P are related by at most 2 of the partial orders $<_i$ (*i.e.*, there is no triangle whose sides have different "colours").

Lemma 4.2. *If a connected poset P satisfies the N-condition, then any two vertices of P have an upper or lower bound.*

Proof. Let $u, v \in P$, $u \neq v$. By assumption, there is a finite sequence $v_0, \ldots, v_k$ of vertices of P with $u = v_0$, $v = v_k$ and such that $v_i < v_{i+1}$ or $v_{i+1} > v_i$ for all $i = 0, \ldots, k-1$. When $k = 1$, it is clear that u and v have both an upper and a lower bound. We proceed by induction on k. Assume that $k > 1$. By the induction hypothesis, u and v_{k-1} have an upper bound or a lower bound. Suppose that x is an upper bound. If $v_{k-1} \geq v$, then x is also an upper bound of u and v. So let $v_{k-1} < v$. If either $x < v$ or $x > v$ or $u < v_{k-1}$ or $u > v_{k-1}$, then it is clear that u and v have an upper or lower bound. If none of these order relations holds, then by the N-condition, $u < v$. The case when u and v_{k-1} have a lower bound is symmetric. $\square$

A graph-theoretic characterization of finite constructible n-posets was given in [2].

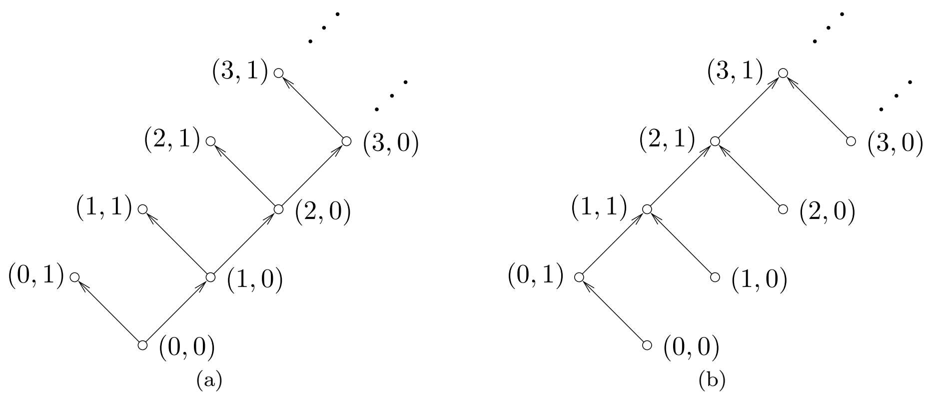

Figure 1: An upward comb (a) and a downward comb (b).

Theorem 4.3 [2]**.** *A finite nonempty n-poset P is in $\mathbf{SP}_{n,F}(\Sigma)$ iff the following hold:*

i) P is complete.
ii) P satisfies the triangle condition.
iii) $(P, <_i)$ is N-free for each $i \in [n]$.

We now set out to provide a related characterization of infinite constructible n-posets.

An *upward comb* is a poset which is isomorphic to the poset whose vertices are the ordered pairs (i, j), for all nonnegative integers $i \geq 0$ and $j \in \{0, 1\}$. Moreover, the immediate successors of a vertex $(i, 0)$ are $(i + 1, 0)$ and $(i, 1)$, and vertices of the form $(i, 1)$ are maximal. The order relation is the transitive closure of the immediate successor relation. We call a poset a *downward comb* if it is isomorphic to the poset whose vertices are the ordered pairs (i, j), where i is a nonnegative integer and j is 0 or 1. The only immediate successor of a vertex $(i, 1)$ is $(i+1, 1)$, and the only immediate successor of vertex $(i, 0)$ is $(i, 1)$. (Thus all vertices of the form $(i, 0)$ are minimal.) The order relation is again the transitive closure of the immediate successor relation.

We say that an n-poset $(P, <_1, \ldots, <_n)$ is *free of upward combs* or *free of downward combs* if none of the posets $(P, <_i)$ contains an (induced) subposet isomorphic to an upward comb or a downward comb, respectively.

Proposition 4.4. *The following conditions hold for all n-posets in $\omega\mathbf{SP}_{n,I}(\Sigma)$:*

i) P is complete;
ii) P satisfies the triangle condition;
iii) for each $i \in [n]$ the poset $(P, <_i)$ satisfies the following conditions:
  a) $(P, <_i)$ is N-free;
  b) $(P, <_i)$ is free of upward combs;

c) $(P,<_i)$ is free of downward combs; and
d) each principal ideal of $(P,<_i)$ is finite.

Proof. By induction on the rank of P. When the rank is 0, one uses Theorem 4.3. □

Remark 4.5. Suppose that $(P,<_1,\ldots,<_n)$ is a complete n-poset satisfying the triangle condition. If $u <_i v <_j w$ holds for some $u,v,w \in P$ and $i,j \in [n]$, then it follows that either $u <_i w$ or $u <_j w$. Thus, the relation $< = <_1 \cup \ldots \cup <_n$ is a linear order on P. By Theorem 4.3 and Proposition 4.4, this gives another proof of Lemma 2.3.

Suppose that $(P,<_1,\ldots,<_n)$ is a complete n-poset satisfying the triangle condition such that each principal ideal of any $(P,<_i)$ is finite and free of upward combs. In Lemma 4.10, we will give several equivalent conditions that for some i, $(P,<_i)$ contains a downward comb. Lemma 4.10 will in turn be applied in the proof of the main characterization theorem. But first we need some auxiliary lemmas.

Lemma 4.6. *Suppose that $(P,<_1,\ldots,<_n)$ is a complete n-poset satisfying the triangle condition and $i_0 \in [n]$. Let $Q_1,Q_2,\ldots,Q_\gamma,\ldots$ be all the connected components of the poset $(P,<_{i_0})$. For any two components Q_{γ_1}, Q_{γ_2}, $\gamma_1 \neq \gamma_2$, there exists an $i \in [n]$, $i \neq i_0$, such that either $x <_i y$ for all $x \in Q_{\gamma_1}$ and $y \in Q_{\gamma_2}$, or $x >_i y$ for all $x \in Q_{\gamma_1}$ and $y \in Q_{\gamma_2}$.*

Proof. Let Q_{γ_1} and Q_{γ_2} be two connected components of $(P,<_{i_0})$. Suppose that $x <_i y$ for some $x \in Q_{\gamma_1}$, $y \in Q_{\gamma_2}$ and $i \in [n]$, $i \neq i_0$. We claim that $x' <_i y'$ for all $x' \in Q_{\gamma_1}$, $y' \in Q_{\gamma_2}$. We only show that $x <_i y$ implies $x' <_i y$ for all $x' \in Q_{\gamma_1}$. (The proof of the fact that $x' <_i y$ implies $x' <_i y'$ for all $y' \in Q_{\gamma_2}$ is symmetrical.) Since $x' \in Q_{\gamma_1}$ means that x and x' are in the same connected component of $(P,<_{i_0})$, there exists a sequence $x_0 = x, x_1, \ldots, x_t = x'$ $(t \geq 0)$, such that $x_{s-1} <_{i_0} x_s$ or $x_{s-1} >_{i_0} x_s$ holds for all $s \in [t]$. We prove by induction on s that $x_s <_i y$ for all s with $0 \leq s \leq t$. Indeed, $x_0 = x <_i y$ holds by assumption. If we have $x_s <_i y$ for some s, $(0 \leq s \leq t-1)$, then since P is complete and satisfies the triangle condition, and by the fact that y is not in the connected component Q_{γ_1}, we have that x_{s+1} and y are related by $<_i$. But $y <_i x_{s+1}$ is not possible, because $x_s <_i y$ and $y <_i s_{s+1}$ would imply $x_s <_i x_{s+1}$, so necessarily $x_{s+1} <_i y$. We conclude that $x_s <_i y$ for all s with $0 \leq s \leq t$. In particular, $x' = x_t <_i y$, as claimed. □

Lemma 4.7. *Suppose that a complete n-poset $(P,<_1,\ldots,<_n)$ satisfies the triangle condition. Then there exists some i such that $(P,<_i)$ is connected.*

Proof. By induction on n. The base case $n=1$ is obvious. Assume $n > 1$. Let P_0, P_1, …, P_γ, … be all the connected components of $(P,<_n)$. If there is only one component then we are finished with the proof. Otherwise, by Lemma 4.6, for all $\gamma_1 \neq \gamma_2$ there exists $j \in [n]$, $j \neq n$ such that $x <_j y$ for all $x \in P_{\gamma_1}$ and $y \in P_{\gamma_2}$, or $y <_j x$ for all $x \in P_{\gamma_1}$ and $y \in P_{\gamma_2}$. Thus, if we collapse each P_γ into a single point, we get a non-singleton $(n-1)$-poset (the relation $<_n$ does not appear) that is also

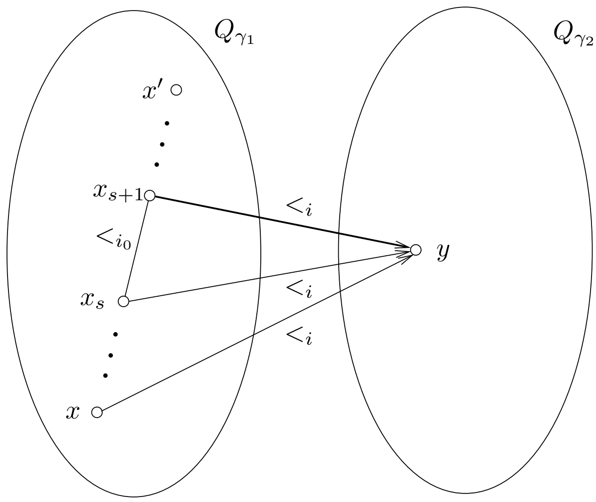

Figure 2: Either $x_s <_{i_0} x_{s+1}$ or $x_s >_{i_0} x_{s+1}$ holds, $x_s <_i y$ implies $x_{s+1} <_i y$ in the proof of Lemma 4.7.

complete and satisfies the triangle condition. Thus, we can apply the induction hypothesis to conclude that there exists some i such that $(P, <_i)$ is connected. $\square$

Lemma 4.8. *Suppose that $(P, <_1, \ldots, <_n)$ is a complete n-poset satisfying the triangle condition and such that for each $j \in [n]$, any principal ideal of $(P, <_j)$ is finite. If for some $i \in [n]$, the poset $(P, <_i)$ contains an ω-chain, then there are only a finite number of elements of P not related to any element of this chain by the relation $<_i$.*

Proof. Let $x_0, x_1, \ldots$ denote an ω-chain in $(P, <_i)$ and assume to the contrary that there are an infinite number of elements $y_0, y_1, \ldots$ not related to any x_k by the relation $<_i$. We claim that for each y_m there is some $j \in [n]$, $j \neq i$, such that $y_m <_j x_k$ holds for all x_k. Indeed, since P is complete and for each $j \in [n]$, any principal ideal of $(P, <_j)$ is finite, we have that $y_m <_j x_{k_0}$ for some j and x_{k_0}. Now $x_0, x_1, \ldots$ all belong to the same connected component of $(P, <_i)$. Thus, by Lemma 4.6, we have $y_m <_j x_k$ for each x_k, proving the claim. Since there are only a finite number of order relations, there must be a $j \in [n]$, $j \neq i$ such that infinitely many of the y_m are below each x_k with respect to the relation $<_j$. But this contradicts Lemma 4.1. $\square$

Lemma 4.9. *Under the assumptions of Lemma 4.8, if $X \subseteq P$ contains an ω-chain in the poset $(P, <_i)$, then either $(P, <_i)$ contains an upward comb, or it is possible to remove a finite number of elements from X such that the resulting set generates a directed filter of $(P, <_i)$.*

Proof. Assume that X contains an ω-chain $x_0, x_1, \ldots$ There are two cases, either $(P, <_i)$ has a finite or an infinite number of maximal elements.

Assume first that there are a finite number of maximal elements. Then the number of elements below some maximal element is also finite. We claim that the infinite set Y obtained from X by removing all such elements generates a directed filter F. Suppose that u, v are distinct elements in F. Since F has no maximal elements, u and v are the least elements of some ω-chains C_1 and C_2 in $(P, <_i)$. But by Lemma 4.8, C_1 and C_2 must have at least one pair of elements related by $<_i$. Thus, u and v have an upper bound in $(P, <_i)$ and thus in F.

Assume now that $(P, <_i)$ has an infinite number of maximal elements. By Lemma 4.8, it follows that for each x_m there is a maximal element y_m with $x_m <_i y_m$. But since each element of P generates a finite ideal of $(P, <_i)$, this is possible only if $(P, <_i)$ contains an upward comb. To see this, let $x_{t_0} = x_0$ and $y_{t_0} = y_0$. When x_{t_s}, y_{t_s} are already defined, let $x_{t_{s+1}}$ be the least x_t such that $x_t \not<_i y_{t_r}$ for all $r \leq s$. Then $x_{t_0}, y_{t_0}, x_{t_1}, y_{t_1}, \ldots$ form an upward comb. □

Lemma 4.10. *Suppose that $(P, <_1, \ldots, <_n)$ is a complete n-poset satisfying the triangle condition. Moreover, suppose that for each $i \in [n]$, any principal ideal of $(P, <_i)$ is finite and $(P, <_i)$ is free of upward combs. Then the following conditions are equivalent.*

- *i) There is a $k \in [n]$ such that $(P, <_k)$ has a directed filter which contains an infinite antichain.*
- *ii) There is a $k \in [n]$ such that $(P, <_k)$ has a directed filter which contains infinitely many minimal elements.*
- *iii) There is a $k \in [n]$ such that $(P, <_k)$ contains a downward comb.*
- *iv) For some $i, j \in [n]$, $i \neq j$, $(P, <_i)$ and $(P, <_j)$ both contain an ω-chain.*

Proof. First, (i) implies (ii). If $(P, <_k)$ has a directed filter containing an infinite antichain, then simply omit the vertices below some element of that antichain. The filter so obtained is directed and contains an infinite number of minimal elements.

Next, we prove that (ii) implies (iii). Suppose that the poset $(P, <_k)$ has a directed filter F which contains infinitely many minimal elements $x_0, x_1, \ldots$ We construct an downward comb in $(P, <_k)$. Let $r_0 = x_0$ and let v_0 be an upper bound of x_0 and x_1 in F, which exists since F is directed. If r_t and v_t are already defined for some $t \geq 0$, then $x_i <_k v_t$ can hold only for finitely many i, otherwise v_t would generate an infinite principal ideal of $(P, <_k)$. Choose an index j such that $x_j <_k v_t$ does not hold and let $r_{t+1} = x_j$ and let v_{t+1} be an upper bound of v_t and x_j in F. It is easy to see that r_0, v_0, r_1, v_1, … form a downward comb.

Now, we prove that (iii) implies (iv). Suppose that $(P, <_i)$ contains a downward comb R. Then the non-minimal elements of R form an ω-chain in $(P, <_i)$. We show that an appropriate subset of the minimal elements of R forms an ω-chain in $(P, <_j)$, for some $j \in [n]$, $j \neq i$. Indeed, any two minimal elements are related by one of the finitely many relations $<_1, <_2, \ldots, <_{i-1}, <_{i+1}, \ldots, <_n$. Thus, according to the Ramsey theorem, *cf.*, *e.g.*, [16], there exists a relation, say $<_j$, ($j \in [n]$, $j \neq i$), and an infinite subset C of the minimal elements of P such that any two elements of C are related by $<_j$. So $(C, <_j)$ is an infinite linearly ordered set, and thus an ω-chain by Lemma 4.1.

Last, we prove that (iv) implies (i). Suppose that for some $i, j \in [n]$, $i \neq j$, both $(P, <_i)$ and $(P, <_j)$ contain an ω-chain, say $x_0, x_1, \ldots$ and $y_0, y_1, \ldots$ Since the two chains can have at most one element in common, we may neglect this element and assume that the two chains are disjoint. Since the set $X = \{x_0, y_0, x_1, y_1, \ldots\}$ contains an ω-chain with respect to the partial order $<_i$, by Lemma 4.9 it is possible to remove a finite number of elements from X such that the resulting set generates a directed filter. Clearly, this filter contains an infinite antichain. □

Remark 4.11. As shown above, under the assumptions of Lemma 4.10, if for some k, $(P, <_k)$ has a directed set containing an infinite antichain, then it has a directed filter containing an infinite antichain.

Remark 4.12. For n-posets $(P, <_1, \ldots, <_n)$ such that no $(P, <_i)$ contains an infinite principal ideal, (iv) of Lemma 4.10 is equivalent to the condition that there is no $i \in [n]$ such that $(P, <_i)$ contains both an infinite ω-chain and an infinite antichain.

In the following theorem, we give a graph-theoretic characterization of infinite constructible n-posets.

Theorem 4.13. *An infinite n-poset P is in $\omega\mathbf{SP}_{n,I}(\Sigma)$ iff the following hold:*

- *i) P is complete;*
- *ii) P satisfies the triangle condition;*
- *iii) for each $i \in [n]$ the poset $(P, <_i)$ satisfies the following conditions:*
  - *a) $(P, <_i)$ is N-free;*
  - *b) $(P, <_i)$ is free of upward combs;*
  - *c) $(P, <_i)$ is free of downward combs; and*
  - *d) each principal ideal of $(P, <_i)$ is finite.*

Remark 4.14. Instead of (iii/c), we could use the negation of any of the equivalent conditions of Lemma 4.10. For later use we note that if an n-poset P satisfies conditions (i)–(iii) of Theorem 4.13, then any (induced) sub-n-poset of P also satisfies these conditions. Moreover, suppose that Q is an n-poset which satisfies (i)–(iii) and no two vertices of Q are related by the relation $<_n$. Then Q is also an $(n-1)$-poset, and as an $(n-1)$-poset, it satisfies conditions (i)–(iii) of Theorem 4.13.

Proof. The necessity part of the Theorem is a restatement of Proposition 4.4. Suppose now that the n-poset P satisfies conditions (i)–(iii). We argue by induction on n to prove that P is in $\omega\mathbf{SP}_{n,I}(\Sigma)$. The base case $n = 1$ is obvious. Assume $n > 1$. We know by Lemma 4.7 that there exists $i_0 \in [n]$ such that the poset $(P, <_{i_0})$ is connected. We distinguish between two cases depending on whether $(P, <_{i_0})$ is directed or not.

Case 1. $(P, <_{i_0})$ is directed. By (iii/d), each element of $(P, <_{i_0})$ is over a minimal element. Since by Lemma 4.10 the set of minimal elements is finite, there is an element strictly over all minimal elements with respect to the partial order $<_{i_0}$.

Let

$$P' = \{\, y \mid \forall x \text{ minimal } x <_{i_0} y\},$$
$$P_0 = P - P'.$$

Note that P_0 and P' are nonempty. We have that $P = P_0 \cdot_{i_0} P'$. Indeed, if $x \in P_0$, $y \in P'$, we show that $x <_{i_0} y$. If x is a minimal element then we are done. Otherwise, let x_0, x_1 be minimal elements such that $x_0 <_{i_0} x$ and $x_1 \not<_{i_0} x$. Since $y \in P'$, we have $x_0 <_{i_0} y$ and $x_1 <_{i_0} y$. It is not possible that $y <_{i_0} x$, since otherwise we would have $x_1 <_{i_0} x$. Thus, if $x \not<_{i_0} y$, then x_0, x_1, x and y form an N in $(P, <_{i_0})$. We conclude that $x <_{i_0} y$, as claimed. Since $P = P_0 \cdot_{i_0} P'$, it follows now by (iii/d) that P_0 is finite.

Now, $(P', <_{i_0})$ is also directed, and P' satisfies conditions (i)−(iii), so that we can decompose P' as $P' = P_1 \cdot_{i_0} P''$, where P'' is the set of all elements of P' strictly over all minimal elements with respect to the partial order $<_{i_0}$, etc. Continuing in the same way, it follows by (iii/d) that

$$P = P_0 \cdot_{i_0} P_1 \cdot_{i_0} \cdots,$$

where each P_i is nonempty, finite, N-free and satisfies the triangle condition. Hence by Theorem 4.3, each P_i belongs to $\mathbf{SP}_{n,F}(\Sigma)$. It follows that P is in $\omega\mathbf{SP}_{n,I}(\Sigma)$.

Case 2. $(P, <_{i_0})$ is not directed. Then there exists some h such that the height $\geq h$ vertices are disconnected in $(P, <_{i_0})$, where the height of a vertex x is the number of elements of the longest chain in $(P, <_{i_0})$ strictly below x. (The height of a vertex is a nonnegative integer by (iii/d).) Indeed, there exist $x, y \in P$ which do not have an upper bound in $(P, <_{i_0})$. Let h denote the minimum of the heights of x and y. Then x and y are disconnected in the subposet of $(P, <_{i_0})$ consisting of the height $\geq h$ vertices, since by Lemma 4.2 they were connected iff they had an upper or lover bound. Assume now that h is minimal such that the vertices of $(P, <_{i_0})$ of height $\geq h$ form a disconnected poset. Since $(P, <_{i_0})$ is connected, we have $h \geq 1$. We show that

$$P = P_0 \cdot_{i_0} P', \tag{9}$$

where

$$P' = \{\, x \mid \text{height}(x) \geq h\},$$
$$P_0 = P - P'.$$

Note that $P_0 \neq \emptyset$. Assume to the contrary that (9) does not hold. Then there exist $x \in P_0$ and $u \in P'$ of height h such that $x <_{i_0} u$ does not hold. By the choice of h, the vertices x and u have an upper bound z. Since P' is disconnected, there is at least one vertex p of height h which belongs to a connected component of P' not containing u, *i.e.*, such that p and u have no upper bound in $(P, <_{i_0})$. But since the vertices of height at least the height of x are connected, it follows

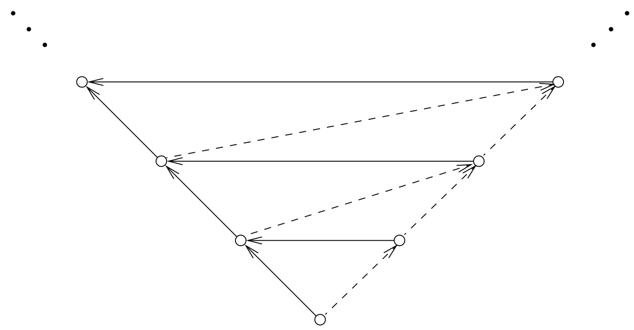

Figure 3: An n-poset that satisfies all conditions of Theorem 4.13 except (iii/c).

from Lemma 4.2 that x and p have an upper bound y. Of course, y belongs to the connected component of $(P', <_{i_0})$ containing p and hence not containing u. Thus, the vertices x, z, u, y form an N, contrary to the assumption (iii/c).

We have thus proved that (9) holds. Note that by (iii/d), P_0 is finite. Thus, by Theorem 4.3, P_0 belongs to $\mathbf{SP}_{n,F}(\Sigma)$. Thus, to complete the proof, we have to show that P' is in $\omega\mathbf{SP}_{n,I}(\Sigma)$. Let $Q_0, Q_1, \ldots$ be all the induced sub-n-posets of P determined by the connected components of $(P', <_{i_0})$. (The list may be finite or infinite.) By Lemma 4.6, if $x <_i y$ holds for some $x \in Q_{j_1}$ and $y \in Q_{j_2}$, where $j_1 \neq j_2$ and $i \neq i_0$, then $x' <_i y'$ holds for all $x' \in Q_{j_1}$ and $y' \in Q_{j_2}$. Let us collapse each Q_j into a single point v_j. Let Q denote the resulting $(n-1)$-poset. Since Q is isomorphic to any subposet R of P' which contains exactly one vertex form each Q_j, it follows using Remark 4.14 that Q satisfies all conditions (i)–(iii). Thus, by induction, Q is in $\omega\mathbf{SP}_{n,I}(\Sigma)$. Moreover, each Q_j is constructible and each Q_j is finite unless v_j is the greatest vertex in Q. Thus, by Lemma 2.5, P' and P also belong to $\omega\mathbf{SP}_{n,I}(\Sigma)$. □

Remark 4.15. Note that the conditions (i), (ii), (iii/a), …, (iii/d) are independent, that is, for each of these conditions there exists an n-poset which satisfies all except that particular condition. In most cases it is almost trivial to construct such an n-poset. For condition (iii/c), Figure 3 shows such an n-poset. An n-poset that violates only condition (iii/b) can be constructed simply by reversing all the relations between the two "chains" of the n-poset in Figure 3.

With a slight modification of the conditions of the previous theorem, we obtain a graph-theoretic characterization of the n-posets in $\mathbf{SP}^{\omega}_{n,I}(\Sigma)$. In the following Theorem, by a *suffix* of an n-poset $(P, <_1, \ldots, <_n)$ we mean a sub-n-poset of P determined by a subset $Q \subseteq P$ which is a filter with respect to all partial orders $<_i$.

Theorem 4.16. *An infinite n-poset P is in* $\mathbf{SP}^{\omega}_{n,I}(\Sigma)$ *iff P satisfies the conditions (i), (ii), (iii/a), (iii/b) and (iii/d) of Theorem* 4.13 *and the following condition:*

(iv) P has, up to isomorphism, a finite number of suffixes.

Proof. It is easy to see by induction on the rank of P that the last condition is necessary. The other conditions are necessary by Theorem 4.13. To prove that the conditions are sufficient, note that (iv) implies (iii/c) of Theorem 4.13. Thus, if P is an infinite n-poset which satisfies the above conditions, then P is constructible. The proof will be complete if we can show that every infinite constructible n-poset satisfying (iv) is strictly constructible. But consider a product $Q = Q_1 \cdot_i Q_2 \cdot_i \ldots$, where each Q_m is in $\mathbf{SP}_{n,F}(\Sigma)$ and $\cdot_i$-irreducible. Consider the sub-n-posets $F_1, F_2, \ldots$ determined by the sets $Q_1 \cup Q_2 \cup \ldots$, $Q_2 \cup Q_3 \cup \ldots, \ldots$, respectively. By (iv), there exist integers j and k, $j < k$, such that F_j is isomorphic to F_k. But then Q_{j+t} is isomorphic to Q_{k+t}, for all $t \geq 0$, proving that Q is $Q_1 \cdot_i \ldots \cdot_i Q_{j-1} \cdot_i (Q_j \cdot_i \ldots \cdot_i Q_{k-1})^{\omega_i}$. It follows now by a straightforward induction on the rank of the infinite constructible n-poset P satisfying (iv) that P is strictly constructible. $\square$

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