ON CAUCHY'S CONVERGENCE TEST

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1. This paper contains two proofs, seeming to be never published, of Cauchy's convergence test, i. e. of the theorem to the effect that the condition

(C) to any positive ε there is a $\varrho = \varrho(\varepsilon)$ such that we have $| a_m - a_n | < \varepsilon$ for every $m, n > \varrho(\varepsilon)$

is necessary and sufficient for the convergence of the infinite sequence $a_1, a_2, \ldots, a_n, \ldots$ Both proofs might be useful for lecture purposes. The first of them has the advantage over the usual class-room proofs of being direct, i. e. not based on the Bolzano-Weierstrass theorem. The second proof furnishes an equivalent form of Cauchy's test allowing some simplifications in Cantor's theory of real numbers, as to be shown in a subsequent paper.

2. Cauchy's convergence test is proved in lectures on Calculus usually by means of the Bolzano-Weierstrass theorem, stating the existence of a convergent subsequence of every bounded infinite sequence. The latter theorem is usually proved by means of successive halving of the interval containing the sequence in question, choosing for the next halving in each case the half, or one of the halves, containing an infinity of terms of the sequence, and considering the common point of the intervals formed thus.

Now, the same idea, slightly modified, leads directly to (the sufficiency[1] of) Cauchy's convergence test. The modification consists in replacing the halves of an interval by two overlapping parts[2], e. g. by its left and right two-thirds. Indeed, let a_n be (the general term of) a sequence satisfying condition (C). We define b_n and c_n recursively as follows. Let $b_1 = a_{\varrho(1)+1} - 1$, $c_1 = a_{\varrho(1)+1} + 1$, and, given b_r and c_r, let $b_r' = \frac{2\,b_r + c_r}{3}$, $c_r' = \frac{b_r + 2\,c_r}{3}$ and

[1] The necessity of condition (C) for the convergence of the sequence a_n is obvious.

[2] The idea of this modification is due to BROUWER who applied, in several publications, overlapping intervals for the construction of numbers the mere existence of which has been proved by means of non-overlapping intervals.

define $b_{r+1} = b_r$, $c_{r+1} = c'_r$ if for all but a finite number of terms of the sequence a_n we have $b_r < a_n < c'_r$, and $b_{r+1} = b'_r$, $c_{r+1} = c_r$ in the opposite case. We prove by induction $b_r < a_n < c_r$ for every r and for every but a finite number of n. Indeed, this is true for $r = 1$ by (C), $\varepsilon = 1$. Suppose it to be true for an r and to be false for $r + 1$. Then, by definition of b_{r+1} and c_{r+1}, we should have $c'_r \leqq a_m < c_r$ and $b_r < a_n \leqq b'_r$ for an infinity of values of m and n, hence $a_m - a_n \geqq c'_r - b'_r$, contrarily to (C), $\varepsilon = c'_r - b'_r$.

Let a be the common point of the intervals $b_r \leqq x \leqq c_r$; then we have $| a_n - a | < c_r - b_r = 2\left(\frac{2}{3}\right)^{r-1}$ for all but a finite number of n, hence $a_n \to a$.

3. We next show that (C) is equivalent to the condition[3]

(C') any two subsequences[4] of a_n differ but in a 0-sequence (i. e. in a sequence converging to 0).

Hence, Cauchy's convergence test is equivalent to the theorem:

Condition (C') is necessary and sufficient for the convergence of the sequence a_n.

Indeed, suppose the sequence a_n satisfies condition (C). Let a_{k_n} and a_{l_n} be any two subsequences of a_n ($k_1 < k_2 < \ldots$, $l_1 < l_2 < \ldots$). Then for any positive ε, we have $| a_{k_n} - a_{l_n} | < \varepsilon$ for $k_n, l_n > \varrho(\varepsilon)$, hence, on account of $k_n, l_n \geqq n$, for $n > \varrho(\varepsilon)$, i. e., we have $a_{k_n} - a_{l_n} \to 0$, thus, a_n satisfies condition (C') too. On the other hand, suppose the sequence a_n does not satisfy condition (C), i. e., for a positive ε_0, and for every positive integer ϱ, there are integers $m = \mu(\varrho)$ and $n = \nu(\varrho)$ such that $m, n > \varrho$ and $| a_m - a_n | \geqq \varepsilon_0$. Define $k_1 = l_1 = 1$ and, given k_r and l_r, let $k_{r+1} = \mu(\max(k_r, l_r))$ and $l_{r+1} = \nu(\max(k_r, l_r))$. Then we have $k_1 < k_2 < \ldots$, $l_1 < l_2 < \ldots$ and $| a_{k_n} - a_{l_n} | \geqq \varepsilon_0$ for every n. Hence $a_{k_n} - a_{l_n}$ does not converge to 0 and the sequence a_n does not satisfy condition (C').

4. Now, we prove the theorem, equivalent to Cauchy's convergence test, formulated in the preceding section. The necessity of condition (C') for the convergence of the sequence a_n being obvious, we have to prove its sufficiency only. First we prove that a sequence a_n satisfying condition (C') is necessarily bounded. Indeed, let a_n be a sequence unbounded from above. Define k_1 as the least k for which we have $a_k > a_1 + 1$ and, given k_r, define k_{r+1} as the least $k > k_r$ for which we have $a_k > a_{r+1} + 1$ (existing, for otherwise $a_{r+1} + 1$ would be an upper bound for the sequence $a_{k_r+1}, a_{k_r+2}, \ldots$,

[3] See also K. KNOPP, *Theorie und Anwendung der unendichen Reihen* (Berlin, 1931) pp. 89–90 (II. Hauptkriterium (3. Form)).

[4] "Subsequence" is meant in this paper so as to include the whole sequence too. Obviously, (C') is equivalent to the condition that a_n differs from any of its subsequences but in a 0-sequence.

and so, the whole sequence a_n would be bounded from above). Then, for $l_n = n$, we have $a_{k_n} > a_{l_n} + 1$ for every n, thus $a_{k_n} - a_{l_n}$ does not converge to 0 and the sequence a_n does not satisfy condition (C'). Similarly, we can show that a sequence unbounded from below does not satisfy condition (C').

Now, let a_n be a sequence satisfying condition (C') and a_{l_n} one of its convergent subsequences existing on account of the Bolzano-Weierstrass theorem. Let $a_{l_n} \to a$; by condition (C'), we have $a_n - a_{l_n} \to 0$; hence, $a_n \to a$.

5. In a lecture where Cauchy's convergence test is based on the proof given in the preceding two sections, it would be advisable, on reasons of style, to give a "qualitative" proof of the Bolzano—Weierstrass theorem too. The following such proof is due to the late Professor Kürschák. It is an instance of a proof by cases of an unusual type[5].

Obviously, it suffices to prove that every infinite sequence has a monotonous subsequence; for if the sequence in question is bounded, the same holds for its subsequences too and so a monotonous subsequence of it is necessarily convergent. Now, let a_n be the sequence in question; define k_1 as the least k for which we have $a_k \leqq a_n$ for every n, and, given k_r, define k_{r+1} as the least $k > k_r$ for which we have $a_k \leqq a_n$ for every $n > k_r$. Of course, k_r may fail to exist; however, if it exists for every r, we have the increasing subsequence a_{k_n} of a_n. If, on the other hand, for a positive integer s, k_s does not exist, then the sequence $a_{k_s+1}, a_{k_s+2}, \ldots$ does not have a least term. In this case, define $l_1 = k_s + 1$ and, given l_r, define l_{r+1} as the least $l > l_r$ for which $a_l < a_{l_r}$ (existing, for otherwise a_{l_r} would be the least term of the sequence $a_{l_r}, a_{l_r+1}, a_{l_r+2}, \ldots$ and thus the sequence $a_{k_s+1}, a_{k_s+2}, \ldots$ formed of the former by subjoining a finite number of terms, would also have a least term[6]). Then a_{l_n} is a decreasing subsequence of a_n.

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[5] This type of the proof by cases consists in giving first a proof method which does not work always and in giving another proof method for the case in which the first method does not work. The proofs for some particular cases, which I know from oral communication of Mr. Bernays, of the arithmetical lemma to which the consistency of Analysis has been reduced by Ackermann (see D. Hilbert, Die Grundlagen der Mathematik, *Abhandlungen aus dem math. Seminar der Hamburgischen Universität*, **6** (1928), pp. 65—85, especially the last paragraph beginning on p. 84), belong to the same type.

[6] As a matter of fact, as easily seen, a_{l_r} would be the least term of the sequence $a_{k_s+1}, a_{k_s+2}, \ldots$ too.

О КРИТЕРИИ СХОДИМОСТИ КОШИ

ЛАСЛО КАЛЬМАР (Сегед)

(Резюме)

Автор дает два новых доказательства критерия *Коши*, относящегося к сходимости бесконечных последовательностей. Первое доказательство аналогично обычному доказательству теоремы *Больцано—Вейерштрасса*, но применяет пересекающие интервалы; оно основывается на том, что если интервал содержит все члены бесконечной последовательности с исключением конечного числа членов, то или левая, или правая $^2/_3$ часть интервала имеет такие же свойства. Второе доказательство основывается на теореме *Больцано—Вейерштрасса* и дает критерий *Коши* в следующей форме: последовательность сходится тогда и только тогда, если от любой своей бесконечной под-последовательности она отличается только последовательностью сходящейся к нулю. Автор сообщает также одно доказательство *Кюрщака* теоремы *Больцано—Вейерштрасса*, которое основывается на том, что каждая последовательность имеет или возрастающую или убывающую под-последовательность.