Decomposition of Weighted Multioperator Tree Automata

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Outline

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1. Motivation
2. Weighted Multioperator Tree Automata
3. Decomposition

Introductory Example

- Ranked alphabet $\Sigma = \{\alpha^{(0)}, \sigma^{(2)}\}$
- Tree transformation:
  - replace every σ by σ_1 or
  - replace every σ by σ_2

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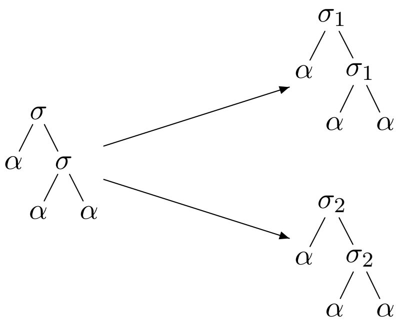
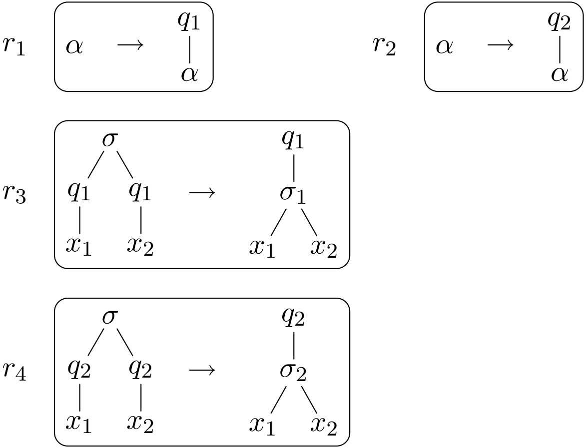


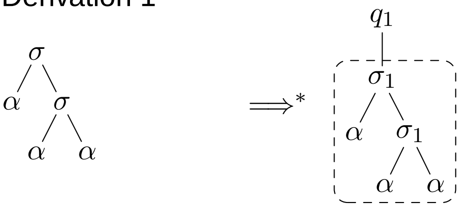
Use a Bottom-Up Tree Transducer

- State set: $\{q_1, q_2\}$, both accepting
- Rule set:

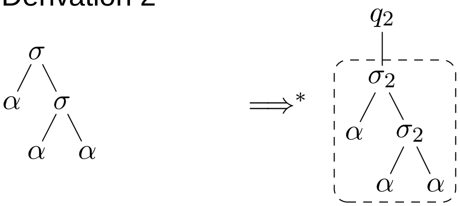


Example Derivation

- Derivation 1



- Derivation 2



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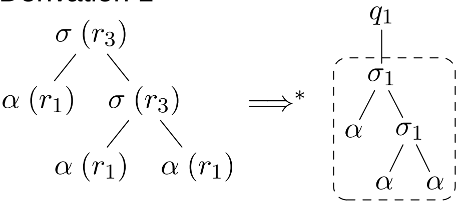

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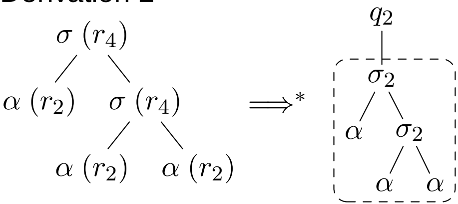

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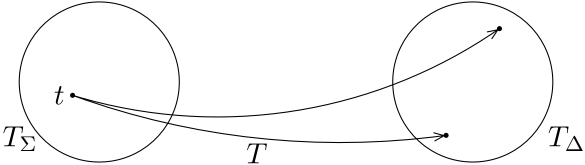

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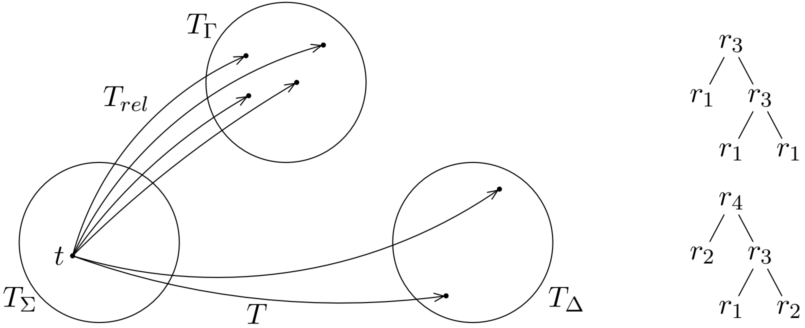

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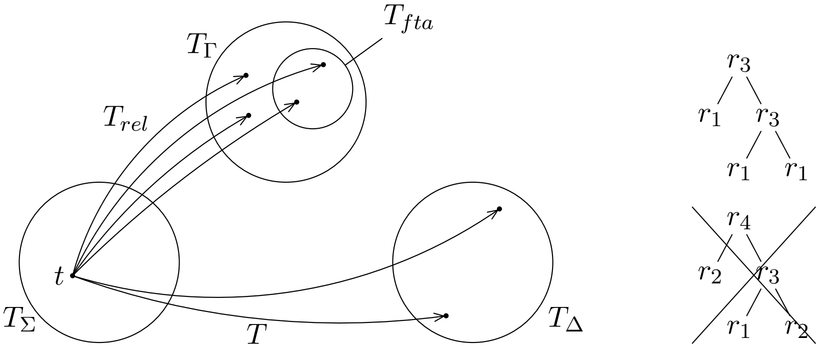

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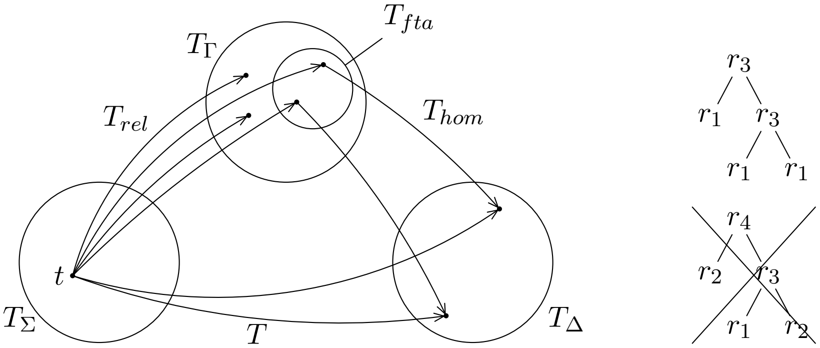

Decomposed Tree Transducer

- Tree Transducer T_{rel}
  - state set: $\{*\}$
  - new output alpabet: $\Gamma = \{r_1, r_2, r_3, r_4\}$
  - $\alpha \to *(r_1)$ $\quad\sigma(*(x_1), *(x_2)) \to *(r_3(x_1, x_2))$
    $\alpha \to *(r_2)$ $\quad\sigma(*(x_1), *(x_2)) \to *(r_4(x_1, x_2))$

- Tree Transducer T_{fta}
  - state set: $\{q_1, q_2\}$
  - $r_1 \to q_1(r_1)$ $\quad r_3(q_1(x_1), q_1(x_2)) \to q_1(r_3(x_1, x_2))$
    $r_2 \to q_2(r_2)$ $\quad r_4(q_2(x_1), q_2(x_2)) \to q_2(r_4(x_1, x_2))$

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History of Results

Theorem (Engelfriet '75)

$BOT_{tt} \subseteq REL_{tt} \circ FTA_{tt} \circ HOM_{tt}$

$BOT_{tt} \supseteq REL_{tt} \circ FTA_{tt} \circ HOM_{tt}$

- Nivat '68
  Decomposition of generalized sequential machines
- Engelfriet, Fülöp, Vogler '02
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Prove a similar result for weighted multioperator tree automata!

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Basic Notions

- ranked alphabet (Σ, rk_Σ)
- let Σ ranked alphabet
  - trees over Σ: T_Σ
  - $\Sigma = \{\alpha^{(0)}, \sigma^{(2)}\}$
    $\rightsquigarrow T_\Sigma = \{\alpha, \sigma(\alpha, \alpha), \sigma(\alpha, \sigma(\alpha, \alpha)), \ldots\}$
- let $t \in T_\Sigma$
  - $pos(t) \subseteq \mathbb{N}^*$
  - $t = \sigma(\alpha, \sigma(\alpha, \alpha))$
    $\rightsquigarrow pos(t) = \{\varepsilon, 1, 2, 21, 22\}$
- let $p \in pos(t)$
  - label $t(p)$
  - $p = 21 \rightsquigarrow t(p) = \alpha$

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Multioperator Monoids

Definition ($Ops(A)$)

Let A be a set, k natural number.

- $Ops^k(A)$: k-ary operations on A,
- $Ops(A)$: operations on A.

Definition (Multioperator monoid)

A multioperator monoid (M-monoid) is a tuple $\underline{A} = (A, \oplus, 0_A, \Omega)$, where

- $(A, \oplus, 0_A)$ is a commutative monoid,
- $\Omega \subseteq Ops(A)$.

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Top Concatentation

Let Σ ranked alphabet, $k \in \mathbb{N}$, $\sigma \in \Sigma^{(k)}$

Definition (Top Concatenation)

Let $t_1, \ldots, t_k \in T_\Sigma$.
$\bar{\sigma}(t_1, \ldots, t_k) = \sigma(t_1, \ldots, t_k)$.

Definition (Language Top Concatenation)

Let $L_1, \ldots, L_k \subseteq T_\Sigma$.
$\bar{\sigma}^L(L_1, \ldots, L_k) = \{\sigma(t_1, \ldots, t_k) | t_i \in L_i\}$.

- Drop the L in $\bar{\sigma}^L$.
- $L_1 = \{\alpha_1, \beta_1\}, L_2 = \{\alpha_2, \beta_2\}$
  $\rightsquigarrow \bar{\sigma}(L_1, L_2) = \{\sigma(\alpha_1, \alpha_2), \sigma(\beta_1, \alpha_2), \sigma(\alpha_1, \beta_2), \sigma(\beta_1, \beta_2)\}$

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Bottom-Up Weighted Multioperator Tree Automata

Definition (Bottom-Up Weighted Multioperator Tree Automaton)

$M = (Q, \Sigma, \underline{A}, \mu, F)$ bottom-up weighted multioperator tree automaton (wta) iff

- Q finite set (of *states*)
- Σ ranked alphabet
- $\underline{A} = (A, \oplus, 0_A, \Omega)$ M-monoid
- $\mu = (\mu_k)_{k \in \mathbb{N}}$ with $\mu_k : \Sigma^k \to (\Omega^{(k)})^{Q^k \times Q}$
- $F \in (\Omega^{(1)})^Q$

Tree Series

Definition (Tree Series)

Let Σ ranked alphabet, $\underline{A} = (A, \oplus, 0_A, \Omega)$ M-monoid.
A tree series over T_Σ and A is a mapping $\varphi : T_\Sigma \to A$.
Notation:

- (φ, s) denotes $\varphi(s)$,
- $A\langle\langle T_\Sigma \rangle\rangle$ set of tree series over T_Σ and A.

Semantics (1)

Let $(Q, \Sigma, \underline{A}, \mu, F)$ be wta, $t \in T_\Sigma$.

Definition (Run)

A run of M on t is a mapping $r : pos(t) \to Q$.
$R_M(t)$ set of all runs of M on t.

Definition (Weight of a run)

Let $r \in R_M(t)$. Define weight mapping $C_r : pos(t) \to A$:

$$C_r(p) = \mu_k(t(p))_{r(p1)\cdots r(pk), r(p)}(C_r(p1), \ldots, C_r(pk)),$$

where $k = rk(t(p))$.

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where $k = rk(t(p))$.

Semantics(2)

Let $M = (Q, \Sigma, \underline{A}, \mu, F)$ be wta

Definition (Run Semantics)

Tree series $\varphi_M \in A\langle\langle T_\Sigma \rangle\rangle$ recognized by M

$$(\varphi_M, t) = \bigoplus_{r \in R_M(t)} F_{r(\varepsilon)}(C_r(\varepsilon)) \; .$$

Example

Example

$M_{ex} = (Q, \Sigma, \underline{A}, \mu, F)$ with

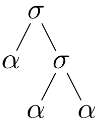
- $Q = \{q_1, q_2\}$,
- $\Sigma = \{\alpha^{(0)}, \sigma^{(2)}\}$,
- $\underline{A} = (\mathcal{P}(T_\Delta), \cup, \emptyset, \{id_{\mathcal{P}(T_\Delta)}, \widetilde{\emptyset}^{(2)}, r_1, r_2, r_3, r_4\})$,
  - $\Delta = \{\alpha^{(0)}, \sigma_1^{(2)}, \sigma_2^{(2)}\}$
  - $r_1 = r_2$ is the constant $\{\alpha\}$,
  - $r_3 : (L_1, L_2) \mapsto \overline{\sigma_1}(L_1, L_2)$,
  - $r_4 : (L_1, L_2) \mapsto \overline{\sigma_2}(L_1, L_2)$,

Example (cont'd)

- $\mu = (\mu_k)_{k \in \mathbb{N}}$,
  - $\mu_0(\alpha)_{\varepsilon, q_1} = r_1$
  - $\mu_0(\alpha)_{\varepsilon, q_2} = r_2$,
  - $\mu_2(\sigma)_{q_1 q_1, q_1} = r_3$,
  - $\mu_2(\sigma)_{q_2 q_2, q_2} = r_4$,
  - otherwise $\mu_2(\sigma)_{\_\_,\_} = \widetilde{\emptyset}^{(2)}$
- $F_{q_1} = F_{q_2} = id_{\mathcal{P}(T_\Delta)}$.

Example Computation (1)

- Given input tree:



- Example run r:

- Weight mapping of r:

- Result:

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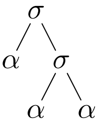

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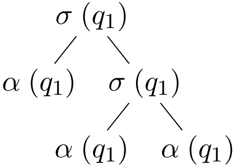

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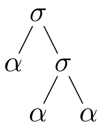

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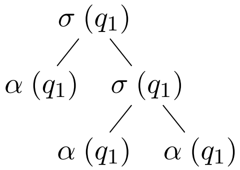

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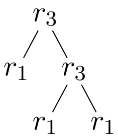

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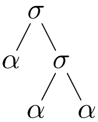

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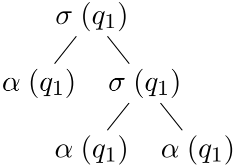

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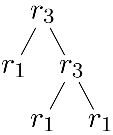

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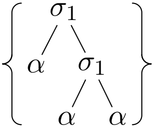
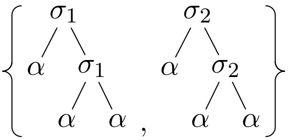


Example Computation (2)

- Summing up over all runs:



Theorem (Maletti '05)

Every bottom-up tree transducer can be simulated by a wta.

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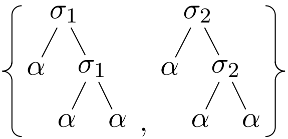

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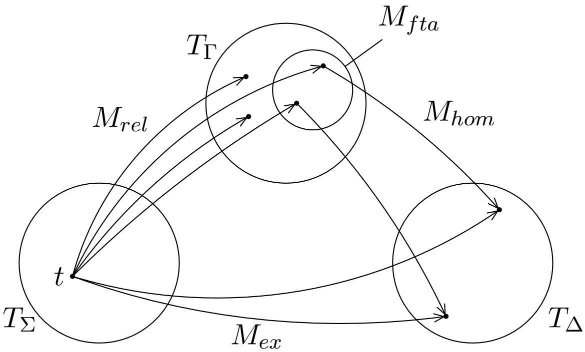
Decomposition of M_{ex}

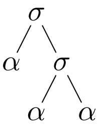


Decomposition (1)

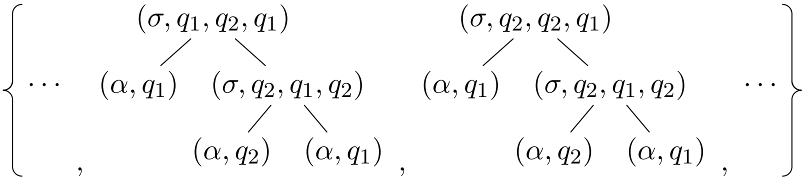
- Define ranked alphabet
  $\Gamma = \{(\alpha, q_1), (\alpha, q_2)\} \cup \{(\sigma, p_1, p_2, q) \mid p_1, p_2, q \in Q\}$.
- Define a relabeling wta

  $M_{rel} = (\{*\}, \Sigma, \underline{A}_{rel}, \mu_{rel}, F_{rel})$ where
  - $\underline{A}_{rel} = (\mathcal{P}(T_\Gamma), \cup, \emptyset, \{id_{\mathcal{P}(T_\Gamma)}, \widetilde{\alpha}, \widetilde{\sigma}\})$
    - $\widetilde{\alpha} = \{(\alpha, q_1), (\alpha, q_2)\}$,
    - $\widetilde{\sigma}(L_1, L_2) = \bigcup_{p_1, p_2, q \in Q} \overline{(\sigma, p_1, p_2, q)}(L_1, L_2)$
  - μ_{rel}:
    - $\mu_{rel}(\alpha)_{\varepsilon,*} = \widetilde{\alpha}$,
    - $\mu_{rel}(\sigma)_{**,*} = \widetilde{\sigma}$,
  - $(F_{rel})_* = id_{\mathcal{P}(T_\Gamma)}$.

Example Computation of M_{rel}



$\rightsquigarrow \quad (M_{rel})$



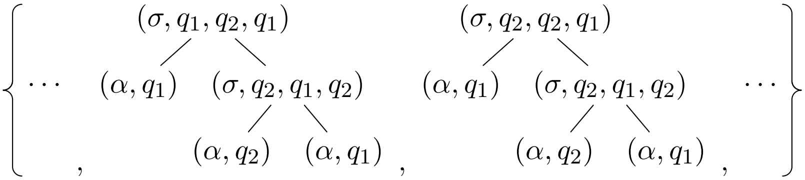
Decomposition (2)

- Define finite state tree automaton

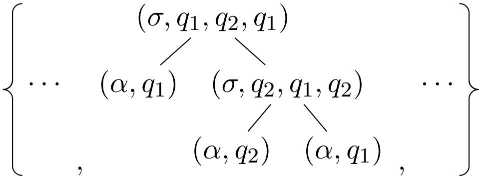
$$M_{fta} = (Q, \Gamma, \underline{A}_{fta}, \mu_{fta}, F_{fta})$$

  - $\underline{A}_{fta} = (\mathcal{P}(T_\Gamma), \cup, \emptyset, \{\widetilde{\emptyset}^{(0)}, \widetilde{\emptyset}^{(2)}, \bar{\alpha}, \bar{\sigma}\})$,
  - μ_{fta}: for every $q, q', p_1, p_1', p_2, p_2' \in Q$:
    - $\mu_{fta}((\alpha, q))_{\varepsilon, q'} = \begin{cases} \bar{\alpha}, & q = q'; \\ \widetilde{\emptyset}^{(0)}, & q \neq q' \end{cases}$
    - $\mu_{fta}((\sigma, p_1, p_2, q))_{p_1' p_2', q'} = \begin{cases} \bar{\sigma}, & q = q' \wedge p_1 = p_1' \wedge p_2 = p_2'; \\ \widetilde{\emptyset}^{(2)}, & \text{otherwise} \end{cases}$
  - $(F_{fta})_q = id_{\mathcal{P}(T_\Gamma)}$ for every $q \in Q$.

Example Computation of M_{fta}



$\rightsquigarrow\ (M_{fta})$



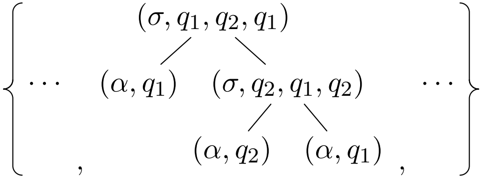
Decomposition (3)

- Define homomorphism

  $M_{hom} = (\{*\}, \Gamma, \underline{A}, \mu_{hom}, F_{hom})$

  - μ_{hom}: for every $q, p_1, p_2 \in Q$:
    - $\mu_{hom}((\alpha, q))_{\varepsilon,*} = \mu(\alpha)_{\varepsilon,q}$,
    - $\mu_{hom}((\sigma, p_1, p_2, q))_{*,*} = \mu(\sigma)_{p_1 p_2, q}$,
  - $(F_{hom})_* = id_A$.

Example Computation of M_{hom}



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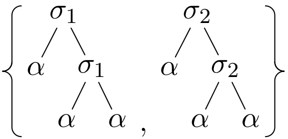

Definition

Let $\underline{A}$ be M-monoid.

- $BOT(\underline{A}) = \{\varphi_M \mid M$ is a wta over $\underline{A}\}$
- $HOM(\underline{A}) = \{\varphi_M \mid M$ is a homomorphism wta over $\underline{A}\}$
- $REL = \{\varphi_M \mid M$ is a relabeling wta$\}$
- $FTA = \{\varphi_M \mid M$ is a fta$\}$
- $BOT^f(\underline{A}) = \{\varphi_M \mid M$ is a fwp wta over $\underline{A}\}$
- $HOM^f(\underline{A}) = \{\varphi_M \mid M$ is a fwp homomorphism wta over $\underline{A}\}$

Definition

A wta $M = (Q, \Sigma, \underline{A}, \mu, F)$ is called final weight preserving (fwp) iff $F_q = id_A$ for every $q \in Q$.

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Summary

Main Result

Theorem (Stüber '06)

For every M-monoid $\underline{A}$:

$$BOT^f(\underline{A}) \subseteq REL \circ FTA \circ HOM^f(\underline{A}) \ .$$

This implies

Theorem (Engelfriet '75)

$BOT_{tt} \subseteq REL_{tt} \circ FTA_{tt} \circ HOM_{tt}$

because of

Theorem (Maletti '05)

Every bottom-up tree transducer can be simulated by a wta.

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Future work

Future work

- For every M-monoid $\underline{A}$
  $BOT^{f}(\underline{A}) \supseteq REL \circ FTA \circ HOM^{f}(\underline{A})$.
- Determine all $\underline{A}$ such that
  $BOT(\underline{A}) = REL \circ FTA \circ HOM(\underline{A})$.

References I

J. Engelfriet.
Bottom-up and top-down tree transformations - a comparison.
Math. Systems Theory, 9(3):198–231, 1975.

Z. Fülöp, A. Maletti, H. Vogler.
Weighted tree automata over multioperator monoids, 2006.

A. Maletti.
Relating tree series transducers and weighted tree automata.
Foundations of Computer Science, 16(4):723–741, 2005.