Fuzzy Set Theory in Image Processing

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Topics of today

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- Is there anything fuzzy about digital images?
- How can we obtain fuzzy images?
  How can we estimate their features?
- What is good about fuzziness in image processing?
- What can we do if, after all, we want crisp images again?

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Btw., what is a set?

"... to be an element..."

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A **set** is a collection of its **members**.

The notion of **fuzzy sets** is an extension of the most fundamental property of sets.

Fuzzy sets allows a grading of **to what extent** an **element** of a set **belongs** to that specific set.

What is a fuzzy set?

A small example

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(Easy! "Yes, or no" ...)

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(Easy! "Yes, or no" ...)

The set F of **big** numbers in X

$$F = \{10, 9, 8, 7, 6, 5, 4, 3, 2, 1\}$$

(Yes or no? ... More like graded ...)

What is a fuzzy set?

Fuzzy is not just another name for probability.

What is a fuzzy set?

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The number 10 is not **probably** big!
...and number 2 is not **probably not** big.

Uncertainty is a consequence of
non-sharp boundaries between the notions/objects,
and not caused by lack of information.

Statistical models deal with random events and outcomes; fuzzy models attempt to capture and quantify nonrandom imprecision.

What is a fuzzy set?

Randomness vs. Fuzziness

Randomness refers to an event that may or may not occur.
Randomness: frequency of car accidents.

Fuzziness refers to the boundary of a set that is not precise.
Fuzziness: seriousness of a car accident.

What is a fuzzy set?

A fuzzy set of **tall men**

Name	Height, cm	Degree of Membership	
		Crisp	*Fuzzy*
Chris	208	1	1.00
Mark	205	1	1.00
John	198	1	0.98
Tom	181	1	0.82
David	179	0	0.78
Mike	172	0	0.24
Bob	167	0	0.15
Steven	158	0	0.06
Bill	155	0	0.01
Peter	152	0	0.00

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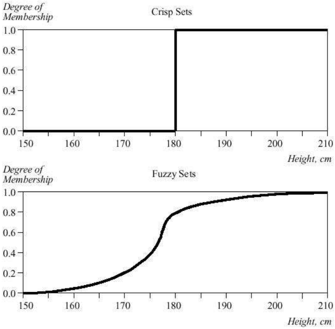
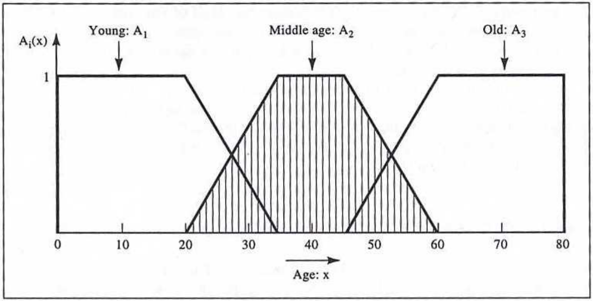


Another example

Age groups



Membership functions representing the concepts of a young, middle-aged, and old person.

Fuzzy sets

The paper

L. A. Zadeh, Fuzzy sets. Information and Control, Vol. 8, pp. 338-353. (1965).

Fuzzy sets

A fuzzy set of a reference set is a set of ordered pairs

$$F = \{\langle x, \mu_F(x)\rangle \mid x \in X\},$$

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To define a fuzzy set $\Leftrightarrow$ To define a membership function

Fuzzy sets

Continuous (analog) fuzzy sets

$$A : X \to [0,1]\,,\ \ X \text{ is dense}$$

Discrete fuzzy sets

$$A : \{x_1, x_2, x_3, \dots, x_s\} \to [0,1]$$

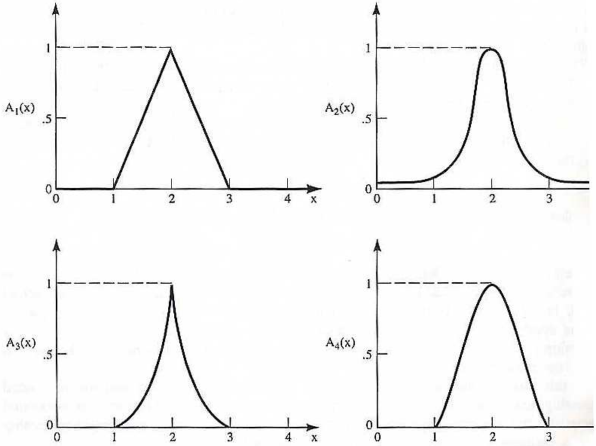
Digital fuzzy sets

If a discrete-universal membership function can take only a finite number $n \geq 2$ of distinct values, then we call this fuzzy set a digital fuzzy set.

$$A : \{x_1, x_2, x_3, \dots, x_s\} \to \{0, \tfrac{1}{n-1}, \tfrac{2}{n-1}, \tfrac{3}{n-1}, \dots, \tfrac{n-2}{n-1}, 1\}$$

Fuzzy sets

Membership functions



Examples of membership functions that may be used in different contexts for characterizing fuzzy sets of real numbers close to 2.

Fuzzy sets

Basic concepts and terminology

The **support** of a fuzzy set A in the universal set X is a crisp set that contains all the elements of X that have nonzero membership values in A, that is,

$$\text{supp}(A) = \{x \in X \mid A(x) > 0\}$$

The **height**, $h(A)$ of a fuzzy set A is the largest membership value attained by any point. If the height of a fuzzy set is **equal to one**, it is called a **normal** fuzzy set, otherwise it is **subnormal**.

Fuzzy sets

Basic concepts and terminology

An **α-cut** of a fuzzy set A is a **crisp set** ${}^{\alpha}A$ that contains all the elements in X that have membership value in A greater than or equal to α.

$$^{\alpha}A = \{x \mid A(x) \geq \alpha\}$$

The 1-cut ${}^{1}A$ is often called the **core** of A.
Note! Sometimes the highest non-empty α-cut ${}^{h(A)}A$ is called the core of A. (in the case of subnormal fuzzy sets, this is different).

The word **kernel** is also used for both of the above definitions.

Fuzzy set operations

Standard set operations

For fuzzy sets A, B on a reference set X, given by the corresponding membership functions $A(x)$ and $B(x)$, standard set operations are:

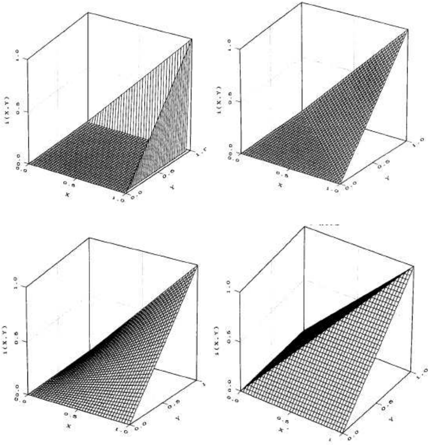
$$\begin{array}{lcl} \bar{A}(x) = 1 - A(x) & - & \text{fuzzy complement} \\ (A \cap B)(x) = \min[A(x), B(x)] & - & \text{fuzzy intersection} \\ (A \cup B)(x) = \max[A(x), B(x)] & - & \text{fuzzy union} \end{array}$$

for all $x \in X$.

Note: There are infinitely many different fuzzy complements, fuzzy intersections, and fuzzy unions!

Fuzzy intersections

Examples of t-norms frequently used



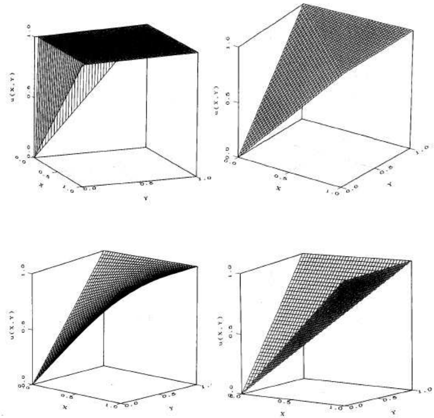
- **Drastic intersection**

$$i(a,b) = \begin{cases} a & \text{if } b = 1 \\ b & \text{if } a = 1 \\ 0 & \text{otherwise} \end{cases}$$

- **Bounded difference**
$i(a,b) = \max[0, a + b - 1]$
- **Algebraic product**
$i(a,b) = ab$
- **Standard intersection**
$i(a,b) = \min[a,b]$

Fuzzy unions

Examples of t-conorms frequently used



- **Drastic union**

  $$u(a,b) = \begin{cases} a & \text{if } b = 0 \\ b & \text{if } a = 0 \\ 1 & \text{otherwise} \end{cases}$$

- **Bounded sum**

  $u(a,b) = \min[1, a+b]$

- **Algebraic sum**

  $u(a,b) = a + b - ab$

- **Standard union**

  $u(a,b) = \max[a,b]$

Duality of fuzzy set operations

Examples of dual triples

Dual triples with respect to the **standard** fuzzy complement

$$\langle \min(a,b), \max(a,b), c_s \rangle$$

$$\langle ab, a+b-ab, c_s \rangle$$

$$\langle \max(0, a+b-1), \min(1, a+b), c_s \rangle$$

$$\langle i_{\min}(a,b), u_{\max}(a,b), c_s \rangle$$

Fuzzy sets in image processing

Important questions to answer:

- How to get fuzzy images?
- How to use fuzziness in image processing?

Fuzzy sets in image processing

A bit more than forty years ago (1965), fuzzy sets were introduced by Zadeh. 14 years later (1979), Rosenfeld introduced fuzzy sets into image analysis [1]. The results obtained in the first five year period (1979-1984), are reported in (Rosenfeld 1984) [2]; various definitions, methods for measuring geometrical and other properties and relationships related to regions in an image defined as fuzzy sets, are summarized.

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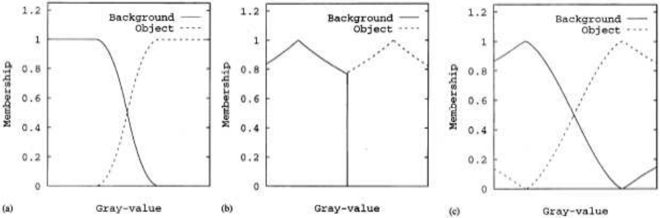
[1] A. Rosenfeld., Fuzzy digital topology. Information and Control, 1979

[2] A. Rosenfeld., The fuzzy geometry of image subsets. Pat. Rec. Letters, 1984.

Fuzzy sets in image processing

Fuzzy thresholding



Membership distributions assigned using

a) Pal and Rosenfeld (1988)
b) Huang and Wang (1995)
c) Fuzzy c-means (Bezdek 1981) algorithms.

Fuzzy sets in image processing

The fuzzy shape analysis techniques addressed in (Rosenfeld 1984) are:

- Connectedness and surroundedness;
- Adjacency;
- Convexity and starshapedness;
- Area, perimeter, and compactness;
- Extent and diameter;
- Shrinking and expanding, medial axes, elongatedness, and thinning;
- Grey-level-dependent properties; splitting and merging.

Fuzzy sets in image processing

The fact that (Rosenfeld 1984) is, during the last twenty years, still one of the main references in most of the papers dealing with fuzzy shapes, indicates not only its outstanding significance and quality, but also the lack of research and results in the field since then.

However, during the last ten years, there has been a steady increase in interest for fuzzy techniques in image processing...

Geometry and shape of spatial fuzzy sets

Basic fuzzy shapes

Most definitions rely on cutworthiness.

- A **fuzzy straight/curved line** is a fuzzy set for which any α-cut, $\alpha \in (0, 1]$, is either empty, or a straight/curved line.
- A **fuzzy disk** is a fuzzy set whose non-empty α-cuts, for $\alpha \in (0, 1]$, are concentric disks.
- A **fuzzy ellipse** is a fuzzy set whose non-empty α-cuts, for $\alpha \in (0, 1]$, are ellipses with the same center, orientation, and eccentricity.

Scalar descriptors of fuzzy sets

Aggregating over α-cuts

Given a function $f : \mathcal{P}(X) \to \mathbb{R}$.

We can extends this function to $f : \mathcal{F}(X) \to \mathbb{R}$, using one of the following equations

$$f(A) = \int_0^1 f(^{\alpha}A)\, d\alpha, \quad (1)$$

$$f(A) = \sup_{\alpha \in (0,1]} [\alpha f(^{\alpha}A)] \quad (2)$$

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Both these definitions provide consistency for the crisp case.

Scalar descriptors of fuzzy sets

Features derived by aggregation over α-cuts

- Area/volume and other geometric moments
- Perimeter/surface area
- Moments (moment invariants)

Area of a fuzzy set

The **area** of a fuzzy set A on $X \subseteq \mathbb{R}$ is

$$\begin{aligned} \mathrm{area}(A) &= \int_X A(x)\, dx \\ &= \int_0^1 \mathrm{area}({}^{\alpha}A)\, d\alpha \end{aligned}$$

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or, for a digital fuzzy set A on $X \subseteq \mathbb{Z}$, with level set $\Lambda(A) = \{\alpha_1, \alpha_2 \ldots \alpha_n\}$

$$\text{area}(A) = |A| = \sum_X A(x) = \sum_{i=1}^{n} (\alpha_i - \alpha_{i-1}) \cdot \text{area}({}^{\alpha}A)\ ,$$

where $\alpha_0 = 0$

Perimeter of a fuzzy set

The **perimeter** of a fuzzy set A

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The **perimeter** of a **fuzzy step set** S given by a piecewise constant membership function μ_S, is defined as

$$\text{perim}(S) = \sum_{\substack{i,j,k \\ i<j}} |\mu_{S_i} - \mu_{S_j}| \cdot |l_{ijk}|,$$

where l_{ijk} is the k^{th} arc along which bounded regions S_i and S_j, defined by (constant-valued) membership functions μ_{S_i} and μ_{S_j}, meet.

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Note:

- Requires estimation of the arc-lengths of the crisp sets.
- Only “horizontal” version.

Moments of a fuzzy set

Moments of a fuzzy set are among the first defined fuzzy concepts.

Geometric moments:

- The **moment** $m_{p,q}(A)$ of a fuzzy set A defined on $X \subset \mathbb{R}^2$, is

$$m_{p,q}(A) = \iint_X A(x,y)\, x^p y^q \, dxdy \; .$$

- The moment $\tilde{m}_{p,q}(A)$ of a digital fuzzy set A on $X \subset \mathbb{Z}^2$, is

$$\tilde{m}_{p,q}(A) = \sum_{(i,j)\in X} A(i,j)\, i^p j^q \; .$$

for integers $p, q \geq 0$.
The moment $m_{p,q}(S)$ has the order $p+q$.

Scalar descriptors of fuzzy sets

Inter-relations

All the definitions listed above reduce to the corresponding customary definitions for crisp sets. However, some inter-relations which these notions satisfy in the crisp case, do not hold for the generalized (fuzzified) definitions.

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For example: The isoperimetric inequality,

$$4\pi \cdot \text{area}(A) \leq \text{perim}^2(A),$$

gives us a way to define a measure of **compactness** of a set A

$$\text{comp}(A) = \frac{4\pi \cdot \text{area}(A)}{\text{perim}^2(A)},$$

Scalar descriptors of fuzzy sets

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This compactness measure, for crisp sets, is the highest for a disk, for which it is equal to one. All other objects have a lower compactness.

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However, for fuzzy sets and definitions given above, the isoperimetric inequality does not hold in general.

In fact, it can be shown that the compactness increases with an increase of fuzziness. A rather unintuitive result in deed.

Inter-relations

Bogomolny proposed (1987) modified definitions, such that they still reduce to their customary crisp counterparts, but the isoperimetric inequality, and also relations between length and area, are fulfilled for a wide class of fuzzy (continuous) sets. However, these definitions are often seen as less intuitive.

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For example: The **perimeter** of a fuzzy set S given by a piecewise constant membership function μ_S, is then defined as

$$\text{perim}(S) = \sum_{\substack{i,j,k \\ i<j}} |\sqrt{\mu_{S_i}} - \sqrt{\mu_{S_j}}| \cdot |A_{ijk}|,$$

where A_{ijk} is the k^{th} arc along which bounded regions S_i and S_j, defined by (constant-valued) membership functions μ_{S_i} and μ_{S_j}, meet.

Scalar descriptors of fuzzy sets
3D

Everything (mentioned so far) generalizes to higher dimensions...

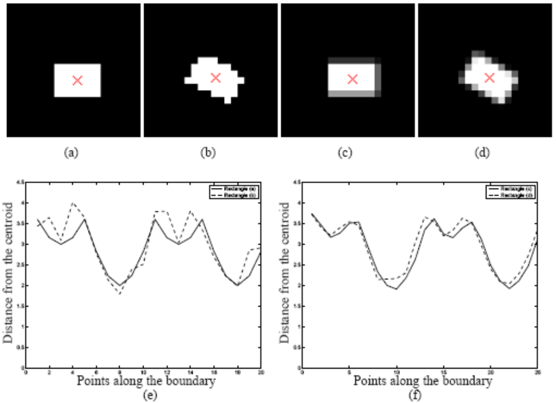
- Volume and higher order moments
- Surface area
- Lengths...

Other types of descriptors

Not only scalar descriptors, but also some vector-valued and non-numerical descriptors are studied for fuzzy sets:

- Signature of a fuzzy set based on the distance from the centroid
- Convexity
- Distances and Distance transforms
- Morphological operations
    - Medial axis transforms and skeletons

Signature of a fuzzy shape: Sensitivity to rotation



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[3] J. Chanussot, I. Nyström, N. Sladoje, Shape Signatures of Fuzzy Sets Based on Distance from the Centroid, PRL, 26(6), pp. 735-746, 2005.

Distances

Set to set distances

Distances between fuzzy sets

a) Membership focused (vertical, co-domain)
b) Spatially focused (horizontal, domain)
c) Mix of spatial and membership (tolerance)
d) Feature distances (low or high dimensional representations)

Distances

Point to point distance

Introduced as grey weighted distances (Rutovitz '68, Levi & Montanari '70) put in a fuzzy framework (Saha '02).

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$$d_A(\pi_i(x, y)) = \int A(t)\, dt$$

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Define the distance along a path π_i between points x and y in the fuzzy set A

$$d_A(\pi_i(x,y)) = \int A(t)\,dt$$

The distance between points x and y in A is the distance along the shortest path

$$d_A(x,y) = \inf_{\pi \in \Pi(x,y)} d_A(\pi)$$

out of all possible paths between x and y, $\Pi(x,y)$.

Distances

Point to point distance

Membership as another dimension
integrate the arc-length

Bloch 1995, Toivanen 1996:

$$d_A(\pi) = \int \sqrt{1 + \left(\frac{dA(t)}{dt}\right)^2}\, dt$$

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Problem: Scale of membership relative to spatial distance

Fuzzy morphologies

- Mathematical morphology is completely based on set theory. Fuzzification started in 1980s.
- Basic morphological operations are **dilation and erosion**. Many others can be derived from them.
- Dilation and erosion are, in crisp case, **dual operations** with respect to the complementation: $D(A) = c(E(cA))$.
- In crisp case, dilation and erosion fulfil a certain number of properties.

How to construct fuzzy mathematical morphology

- Infinitely many fuzzy mathematical morphologies can be constructed.
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- It is desirable to understand the differences and to be able to make choices (of operations, structuring elements,...) in the way that fits the task the best.
- Main construction principles:

  α-cut decomposition;

  fuzzification of set operations.

Estimation of features

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- Discretization causes unavoidable loss of information, therefore recovering of this information cannot be complete.
- **Estimation** of relevant features of the continuous original, based on its discrete representation is, therefore, what remains. Various algorithms are proposed.

Estimation of features

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- Shape descriptors, estimated from fuzzy digital object representations are shown to be **more precise** than those obtained from crisp shape representations.
- Estimation of

  area
  higher order moments
  ... in 2D and 3D
  perimeter and surface area
  projections
  distances

  are some of the features studied by now.

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- It is shown that the precision of estimates increases both with an increase of spatial and of **membership resolution** (number of membership levels - grey levels- used for representation).
- For **pixel coverage** based membership function, estimation error bounds are derived for some descriptors.

Estimation of features

- It is possible to compensate for insufficient spatial resolution by using available grey levels for representing partial memberships!

Estimation of features

- It is possible to compensate for insufficient spatial resolution by using available grey levels for representing partial memberships!
- Only a few membership levels can significantly improve the performance of a descriptor. However, it is noticed that there should always be a balance between these two resolutions, to obtain good results.

Conclusions, Part I

- Spatial fuzzy sets are of a particular interest in image analysis.
- Features of spatial fuzzy sets - shape descriptors.
- “Horizontal” and “vertical” approach in definitions.
- Generalizations are numerous, find the one that suits you.
- High precision estimation - often a particular membership functions considered.
- Spatial and membership resolution - compensate one for another.
- A number of features are generalized and defined for fuzzy shapes, but still many are left.
- Fuzzy values?

Defuzzification

But what if we, after all, want to get back to crisp shapes and traditional image processing tools?

Defuzzifiction

Definition

Defuzzification is a process that maps a fuzzy set to a crisp set.

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Approaches

- Defuzzification to a point.
- Defuzzification to a set.

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Goals

- Generating a good representative of a fuzzy set.
- Recovering a crisp original set.

Defuzzification to a point

Main approaches

- Maxima methods and derivatives

  Selection of an element from the core of a fuzzy set as defuzzification value. Main advantage is simplicity.
- Distribution methods and derivatives

  Conversion of the membership function into a probability distribution, and computation of the expected value. Main advantage is continuity property.
- Area methods

  The defuzzification value divides the area under the membership function in two (more or less) equal parts.

Defuzzification to a set

Main approaches

- α-cuts

  chosen at various levels α.

- Average α-cuts

  based on an integration of set-valued function, called Kudo-Aumann integration.

Defuzzification in image processing

Defuzzification by feature distance minimization