Advanced Optimization

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TUM = Total Unimodular Matrices

Linear algebra

- Determinant of matrix A: $\det(A)$
- It is a scalar value that can be computed from the elements of a square matrix and encodes certain properties of the linear transformation described by the matrix.
- Geometrically, it is the signed volume of the n-dimensional parallelepiped spanned by the column or row vectors of the matrix.
- The determinant is positive or negative according to whether the linear transformation preserves or reverses the orientation of a real vector space.

Linear algebra

$$|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc. \tag{1}$$

$$\begin{aligned} |A| = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} &= a \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \begin{vmatrix} d & e \\ g & h \end{vmatrix} \qquad (2) \\ &= aei + bfg + cdh - ceg - bdi - afh. \qquad (3) \end{aligned}$$

TUM

- Definition. A square matrix U is *unimodular* if $\det(U) = \pm 1$
- Definition. A matrix $M \in \mathbb{R}^{m \times n}$ is called *totally unimodular* if every square non-singular submatrix of M is unimodular.

Put it differently: all submatrix U of M has $\det(U) \in \{0, 1, -1\}$.

TUM - properties: For any TUM matrix M...

- all elements of M are either 0 or 1 or -1.
- $-M$ and M^T is also TU
- **Theorem:** $[M \ \ I]$ is also TU
  Proof (incomplete)
  Let $\mathbf{e}_i = (0, 0, \ldots 1, 0, \ldots, 0)^T$. We are going to show that $M' = [M \ \ \mathbf{e}_i]$ is TU.

  Choose a $k \times k$ submatrix U from M' (k rows and k columns).
  – if the last column and the ith row is included then $\det(U) = \pm 1 \det(M^*)$, where M^* is a submatrix of M
  – if the ith row is not included then $\det(U) = 0$.
  – if the last column is not included, then U is a submatrix of M, which is TUM. $\square$

TUM - integer solution of LP

Theorem. Let $M \in \mathbb{R}^{m \times n}$ (where $m < n$) be full row-rank and totally unimodular. Let $\mathbf{b} \in \mathbb{Z}^m$ and $\mathbf{c} \in \mathbb{R}^n$.
Then the LP:

$$\begin{aligned} & \min \mathbf{c}^T \mathbf{x} \\ \text{subject to:} \quad & M\mathbf{x} = \mathbf{b} \\ & \mathbf{x} \geq 0 \end{aligned}$$

has <u>integer</u> $\mathbf{x}^* \in \mathbb{Z}^n$ solution.

This is important result as we can use any LP solver to get integer solution.
Time of solving LP: polynomial, whereas solving ILP: exponential.

<u>Proof.</u> An optimal solution of an LP is a possible basis (extreme point of the polyhedron $\mathcal{P} = \{M\mathbf{x} = \mathbf{b}, \mathbf{x} \geq 0\}$). We are going to show that these extreme points are integers.

A vector $\mathbf{x}$ is called possible basis if

- $M\mathbf{x} = \mathbf{b}, \mathbf{x} \geq 0$ which means that $\mathbf{x}$ is feasible.
- $\mathbf{x}$ has at most m non-zero elements.
  Let $B(\mathbf{x}) \subset \{1, \ldots, n\}$ be those indices which correspond to the non-zero elements of $\mathbf{x}$.
- The submatrix A of M which is selected by the indicies $B(\mathbf{x})$ is non-singular, i.e., $\det(A) \neq 0$. In this case the system of linear equations $A\hat{\mathbf{x}} = \mathbf{b}$ can be solved, where $\hat{\mathbf{x}}$ is a sub-vector of $\mathbf{x}$ which is selected by $B(\mathbf{x})$.

(repeated from the previous slide):

- The submatrix A of M which is selected by the indicies $B(\mathbf{x})$ is non-singular, i.e., $\det(A) \neq 0$. In this case the system of linear equations $A\hat{\mathbf{x}} = \mathbf{b}$ can be solved, where $\hat{\mathbf{x}}$ is a sub-vector of $\mathbf{x}$ which is selected by $B(\mathbf{x})$.

  Apply Cramer's rule:
  $$\hat{\mathbf{x}}_i = \frac{\det(A_i)}{\det(A)},$$
  where matrix A_i is obtained by changing the ith column in A into $\mathbf{b}$.
  We know that $\mathbf{b}$ is integer.
  $\det(A) = \pm 1$ for sure since matrix A is non-singular and it is a sub-matrix of M.
  $\det(A_i)$ needs to be integer.
  $\Rightarrow \hat{\mathbf{x}}_i$ is integer too $\Rightarrow \mathbf{x}^*$ is integer. $\square$

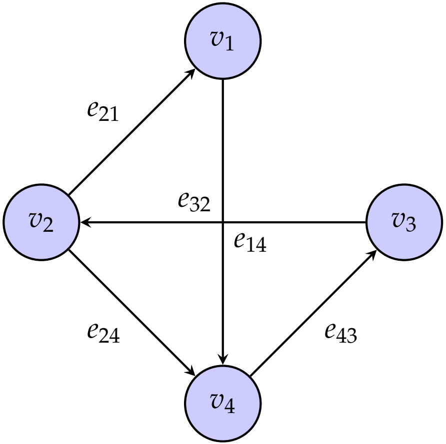
Graphs and TUMs

- Let $G = (V, E)$ be a directed graph.
- Let B the incidence matrix of G.
- B has dimension $|V| \times |E|$ and by definition

$$b_{ij} = \begin{cases} -1 & \text{if node } i \text{ is the tail of edge } j, \\ 1 & \text{if node } i \text{ is the head of edge } j, \\ 0 & \text{otherwise.} \end{cases}$$

- Example.

Example



Directed graph $G(V, E)$

Incidence matrix

	e_{14}	e_{21}	e_{24}	e_{32}	e_{43}
v_1	-1	1	0	0	0
v_2	0	-1	1	0	0
v_3	0	0	0	-1	1
v_4	1	0	-1	1	-1

Graphs and TUMs

- **Theorem.** Matrix B is TU.
- Proof. By induction.
  - Assume that the theorem holds for all sub-matrices of B of size $(k-1) \times (k-1)$.
  - Take a sub-matrix U of size $k \times k$.
  - There are 3 possibilities.
    1) U has all-zero column. $\Rightarrow \det(U) = 0$.
    2) U has a column which contains a non-zero element.
       $\det(U) = \pm 1 \cdot \det(U^*)$, where U^* is a sub-matrix of size $(k-1) \times (k-1)$.

Graphs and TUMs

3) All columns of U has 2 non-zero elements.
   Within a column, one of them is $+1$ and the other one is -1.
   Hence, the sums of the columns are all equal to 0.
   In this case, the rows of the matrix are linearly dependent.
   $\Rightarrow \det(U) = 0$.

Graphs and TUMs

- Sufficient conditions: Let $A = [a_{ij}]$ be a matrix such that

  i) $a_{ij} \in \{+1, -1, 0\}$ for all i, j.

  ii) Each column contains at most two nonzero coefficients,

  $$\sum_{i=1}^{m} |a_{ij}| \leq 2 \quad (j \in [1, n]).$$

  iii) The set M of rows can be partitioned into (M_1, M_2) such that each column j containing *two* nonzero coefficients satisfies

  $$\sum_{i \in M_1} a_{ij} - \sum_{i \in M_2} a_{ij} = 0.$$

  Then A is totally unimodular.

Bipartite graphs and TUMs

- **Theorem.** Let G be a bipartite graph and B^+ its unsigned incidence matrix. Then B^+ is TU.

- Proof. Each column of B^+ contains exactly two nonzero components, a 1 for some $v \in V_1$, and a 1 for some $w \in V_2$.

  Therefore, the sufficient criterion of the above theorem applies for the choice $M_1 = V_1$, $M_2 = V_2$.

TUM - example 01

- Shortest path in directed graph G (from s to t)
- decision variable

$$x_{ij} = \begin{cases} 1 & \text{if edge } (i,j) \text{ is part of the shortest path,} \\ 0 & \text{otherwise.} \end{cases}$$

- LP modell:

$$\min \sum_{(i,j)\in E} x_{i,j}$$

subject to

$$(B\mathbf{x})_i = \begin{cases} -1 & \text{if } i = s, \\ 1 & \text{if } i = t, \\ 0 & \text{otherwise.} \end{cases}$$

Matrix B is the incidence matrix of G.

TUM - example 01

- Another notation:

$$
\begin{aligned}
& \min \mathbb{1}^T \mathbf{x} \\
\text{subject to} \quad & B\mathbf{x} = (-1, 0, 0, \ldots, 0, 1)^T \\
& \mathbf{x} \geq 0.
\end{aligned}
$$

- We do not need to prove that $x_{i,j} \in \{0, 1\}$ as it gets automatically fulfilled.

TUM - example 02

- Maximal pairing in bipartite graphs
- decision variable:

$$x_{ij} = \begin{cases} 1 & \text{if edge } (i, j) \text{ is included in the pairing,} \\ 0 & \text{otherwise.} \end{cases}$$

- LP model

$$\begin{aligned} & \max \mathbb{1}^T \mathbf{x} \\ \text{subject to} \quad & B^+ \mathbf{x} \leq \mathbb{1}, \end{aligned}$$

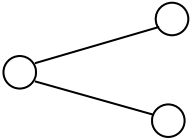
where B^+ is the unsigned incidence matrix of the bipartite graph.

- Since B^+ is TU, it is enough to have

$$\mathbf{x} \geq 0$$

as $x_{ij} \in \{0, 1\}$ holds automatically.

- The meaning of constraint $B^+\mathbf{x} \leq \mathbb{1}$:
in case we have edges as



then either the top one or the bottom one is chosen, but never together.

TUM - example 03

- Minimum $s - t$ cut