MILP: Why do we need integer variables?

- Sometimes, the variables represent things that, because of its nature, can only take integer values: number of books to buy, number of facilities to be located, number of people to be hired.
- Logic constraints can be modeled via binary/integer variables.
- Some nonlinear models can be approximated using MILP.

Modeling tricks

Dealing with unrestricted variables

If $x_1, \ldots, x_r \in \mathbb{R}$ are unrestricted variables, and our algorithm only works with nonnegative variables, we can change:

$$x_i = x_i^1 - x_i^2, \; x_i^1, x_i^2 \geq 0.$$

This duplicates the number of variables. We can do better and introduce only one additional variable x^* which just move all the variable to the right:

$$x_i = x_i^* - x^*, \; x_i^*, x^* \geq 0.$$

Example

$$\begin{aligned} x_1 + x_2 &\leq 1 \\ 2x_1 - x_2 &\geq 3 \\ x_1, x_2 &\in \mathbb{R} \end{aligned} \quad \text{is equivalent to} \quad \begin{aligned} x_1^* - x^* + x_2^* - x^* &\leq 1 \\ 2(x_1^* - x^*) - (x_2^* - x^*) &\geq 3 \\ x_1^*, x_2^*, x^* &\geq 0 \end{aligned}$$

Modeling tricks

Converting linear equalities into linear inequalities

Using slack and surplus variables we can transform inequalities into equalities. But we can also do the opposite.

$$a_i^t x = b_i, i = 1 \ldots, m$$

can be transformed into

$$\begin{gathered} a_i^t x \leq b_i, i = 1 \ldots, m \\ \left(\sum_{i=1}^m a_i^t\right) x \geq \sum_{i=1}^m b_i \end{gathered}$$

Example

$$\begin{aligned} x_1 + x_2 &= 1 \\ 2x_1 - x_2 &= 3 \end{aligned} \quad \text{is equivalent to} \quad \begin{aligned} x_1 + x_2 &\leq 1 \\ 2x_1 - x_2 &\leq 3 \\ 3x_1 &\geq 4 \end{aligned}$$

Modeling tricks

Converting nonlinear objective functions into linear

$$\begin{array}{ll} \min & f(x) \\ s.t. & x \in X \end{array} \quad \text{is equivalent to} \quad \begin{array}{ll} \min & t \\ s.t. & f(x) \le t \\ & x \in X \end{array}$$

Modeling tricks

Dealing with absolute values

$$\begin{array}{ll} \min & \sum_{i=1}^{m} |f_i(x)| \\ s.t. & x \in X \end{array}$$

is equivalent to

$$\begin{array}{lll} \min & \sum_{i=1}^{m} t_i & \\ s.t. & x \in X & \\ & f_i(x) \leq t_i, & i = 1, \dots, m \\ & -f_i(x) \leq t_i, & i = 1, \dots, m \\ & t_i \geq 0, & i = 1, \dots, m \end{array}$$

Modeling tricks

Dealing with the max function

$$\begin{array}{ll} \min & \max_{i=1}^{m}\{f_i(x)\} \\ s.t. & x \in X \end{array}$$

is equivalent to

$$\begin{array}{ll} \min & t \\ s.t. & x \in X \\ & f_i(x) \leq t, i = 1, \ldots, m \end{array}$$

MILP: modeling tricks

Alternative sets of constraints

Consider two set of constraints

$$f_i^1(x) \leq b_i^1, i = 1, \ldots, m_1$$
$$f_i^2(x) \leq b_i^2, i = 1, \ldots, m_2$$

A set of constraints stating that at least one of the two above sets of constraints must be satisfied can be written as

$$f_i^1(x) - \delta_1 M_i^1 \leq b_i^1, i = 1, \ldots, m_1$$
$$f_i^2(x) - \delta_2 M_i^2 \leq b_i^2, i = 1, \ldots, m_2$$
$$\delta_1 + \delta_2 \leq 1$$
$$\delta_1, \delta_2 \in \{0, 1\}$$

provided that the parameters M_i^j satisfy
$f_i^j(x) \leq b_i^j + M_i^j, i = 1, \ldots, m_j, j = 1, 2$

MILP: modeling tricks

Alternative sets of constraints

Consider two set of constraints

$$f_i^1(x) \leq b_i^1, i = 1, \ldots, m_1$$
$$f_i^2(x) \leq b_i^2, i = 1, \ldots, m_2$$

A set of constraints stating that only one set of contraints must be satisfied can be written as

$$f_i^1(x) - \delta_1 M_i^1 \leq b_i^1, i = 1, \ldots, m_1$$
$$f_i^2(x) - \delta_2 M_i^2 \leq b_i^2, i = 1, \ldots, m_2$$
$$\delta_1 + \delta_2 = 1$$
$$\delta_1, \delta_2 \in \{0, 1\}$$

provided that the parameters M_i^j satisfy
$f_i^j(x) \leq b_i^j + M_i^j, i = 1, \ldots, m_j, j = 1, 2$

MILP: modeling tricks

Alternative sets of constraints

Consider two set of constraints

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$$\delta_1 + \delta_2 = 1$$
$$\delta_1, \delta_2 \in \{0, 1\}$$

provided that the parameters M_i^j satisfy
$f_i^j(x) \leq b_i^j + M_i^j, i = 1, \ldots, m_j, j = 1, 2$

This can be used to define nonconvex polygonal feasible sets.

MILP: modeling tricks

Conditional constraints 1

A conditional constraint of the form

$$f(x) > a \Longrightarrow g(x) \leq b$$

can be modeled with the alternative set of constraints

$$f(x) \leq a \quad \text{and/or} \quad g(x) \leq b$$

which in turn can be modeled as explained before (see more equivalences for conditional statements later on).

MILP: modeling tricks

K out of N constraints must hold

If we have a set of N constraints

$$f_1(x) \leq b_1, \ldots, f_N(x) \leq b_N$$

and only K out of the N constraints must hold, this can be modeled as follows:

$$f_1(x) \leq b_1 + M_1\delta_1$$
$$\ldots$$
$$f_N(x) \leq b_N + M_N\delta_N$$
$$\sum_{i=1}^{N} \delta_i = N - K$$
$$\delta_i \in \{0, 1\}, i = 1, \ldots, N$$

where M_i is an upper bound for $f_i(x) - b_i$.

MILP: modeling tricks

Modeling fixed costs

The discontinuous function to be minimized

$$\min f(x) = \begin{cases} 0 & \text{if } x = 0 \\ k + g(x) & \text{if } 0 < x \leq b \end{cases}$$

which sets a fixed cost k in case the variable x is used (in case $x > 0$) can be written as

$$\begin{aligned} \min \quad & k\delta + g(x) \\ s.t. \quad & x \leq b\delta \\ & x \geq 0 \\ & \delta \in \{0, 1\} \end{aligned}$$

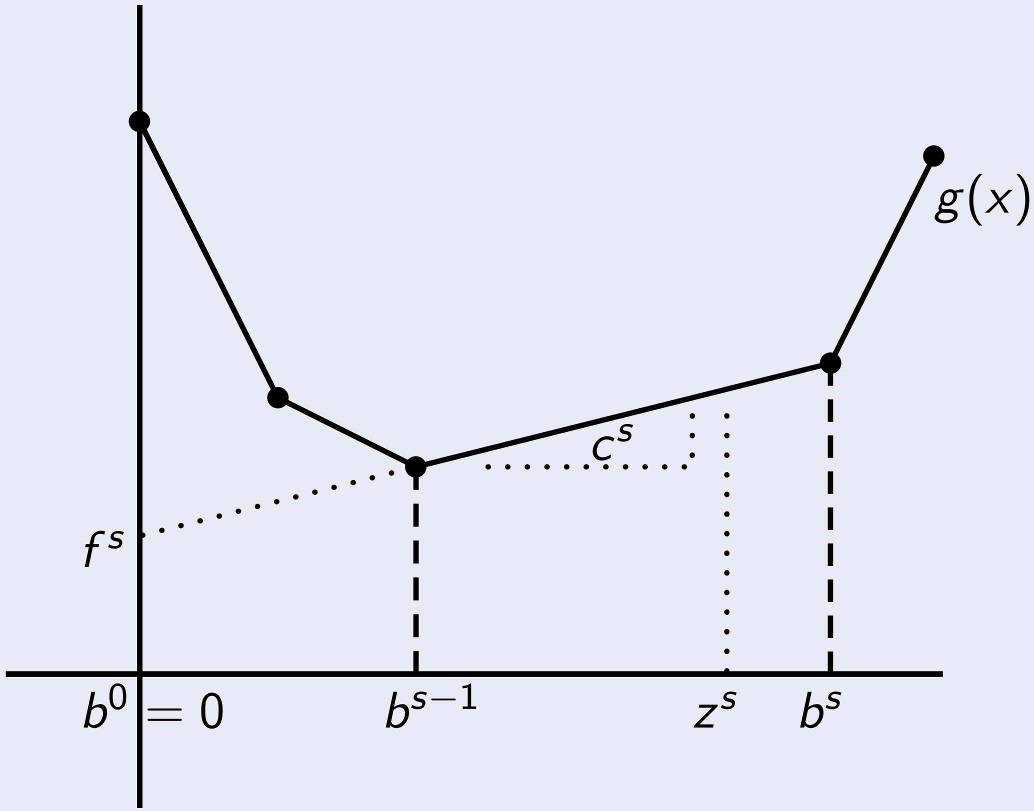
Notice that

$$\delta = \begin{cases} 0 & \text{if } x = 0 \\ 1 & \text{if } x > 0 \end{cases}$$

MILP: modeling tricks

Modeling a piecewise linear function

Consider the piecewise linear function $g(x)$ of the picture. Assume that there are $p+1$ breaking points, $b^0, \dots, b^p$. The slope of the s-th segment $[b^{s-1}, b^s]$ will be denoted by c^s, and the point where the line containing that segment cuts the $0Y$-axis by f^s. Then, the value of $g(x)$ at a point z^s on that segment is given by $g(z^s) = f^s + c^s z^s$.



MILP: modeling tricks

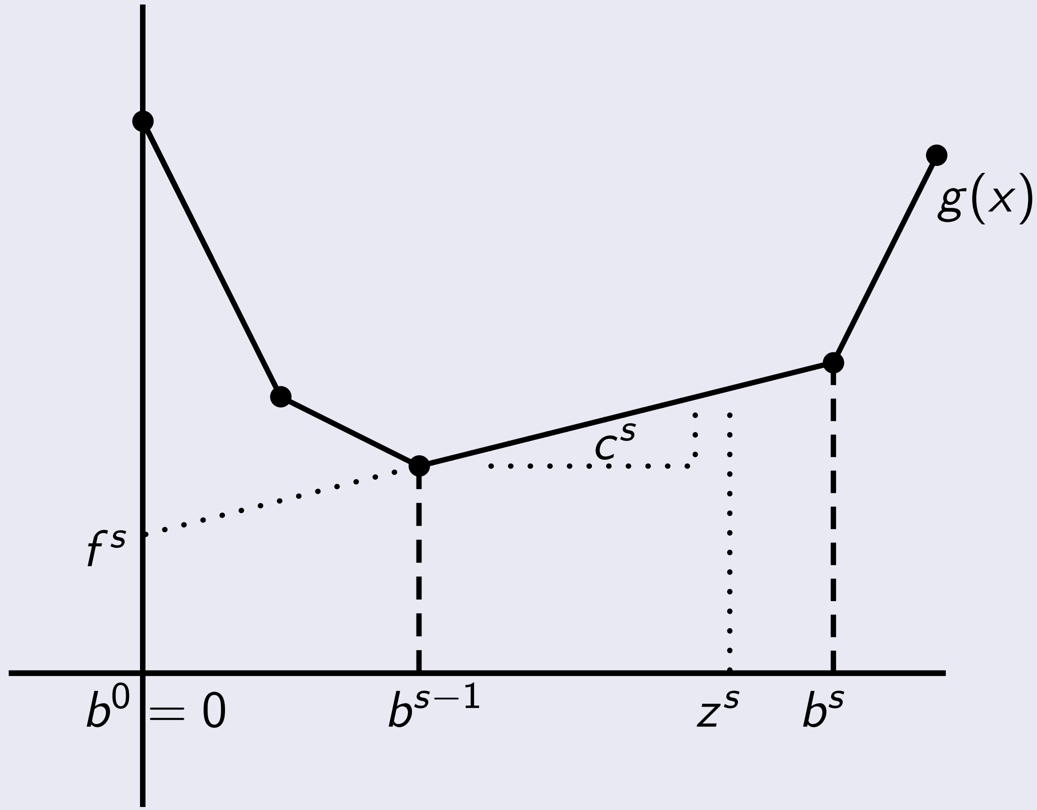
Modeling a piecewise linear function 1

Let us denote

$$z^s = \begin{cases} x & \text{if } x \in [b^{s-1}, b^s] \\ 0 & \text{otherwise} \end{cases} \text{ and } \delta_s = \begin{cases} 1 & \text{if } z^s > 0 \\ 0 & \text{otherwise} \end{cases}, s = 1 \ldots, p$$

Then function $g(x)$ can be rewritten as follows:

$$g(x) = \sum_{s=1}^{p} (c^s z^s + f^s \delta_s)$$
$$x = \sum_{s=1}^{p} z^s$$
$$b^{s-1}\delta_s \le z^s \le b^s \delta_s$$
$$\sum_{s=1}^{p} \delta_s = 1$$
$$\delta_s \in \{0, 1\}, s = 1 \ldots, p$$



MILP: modeling tricks

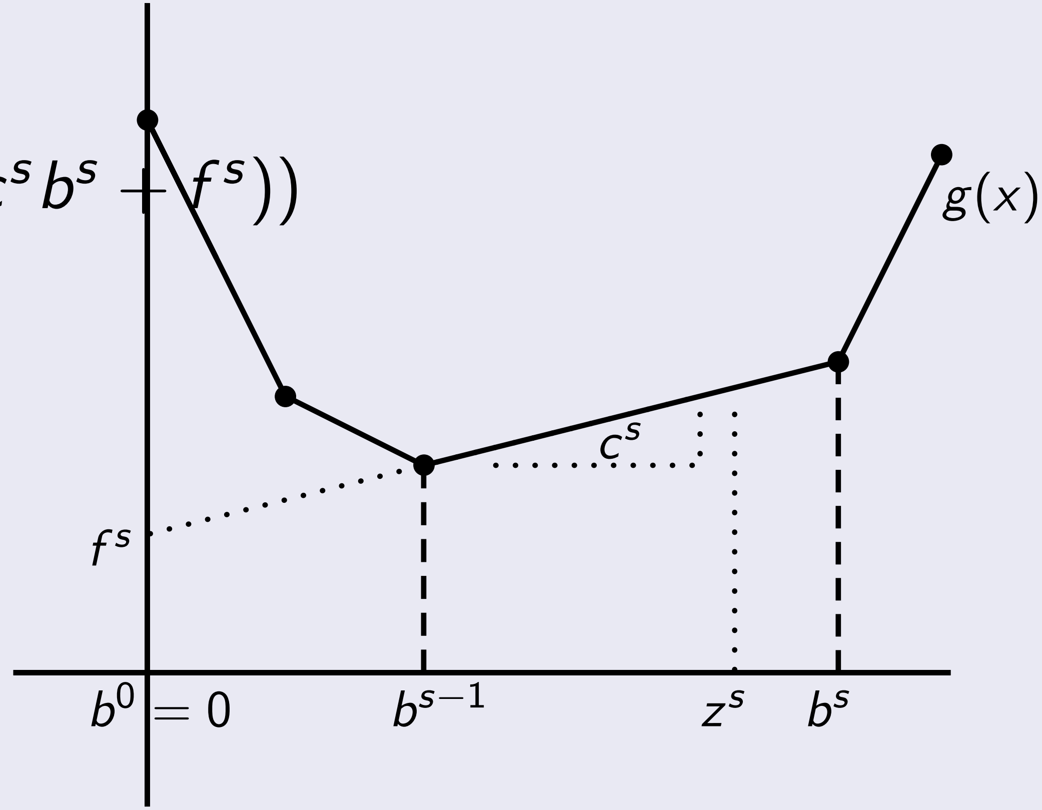
Modeling a piecewise linear function 2

Alternatively, since each point $z^s \in [b^{s-1}, b^s]$ may be written as a convex combination of its end points, $(b^{s-1}, c^s b^{s-1} + f^s)$ and $(b^s, c^s b^s + f^s)$,

$$(z^s, g(z^s)) = \lambda_s(b^{s-1}, c^s b^{s-1} + f^s) + \mu_s(b^s, c^s b^s + f^s), \ \lambda_s + \mu_s = 1$$

we can also rewrite the function $g(x)$ as follows:

$$g(x) = \sum_{s=1}^{p}(\lambda_s(c^s b^{s-1} + f^s) + \mu_s(c^s b^s + f^s))$$
$$x = \sum_{s=1}^{p}(\lambda_s b^{s-1} + \mu_s b^s)$$
$$\lambda_s + \mu_s = \delta_s$$
$$\sum_{s=1}^{p} \delta_s = 1$$
$$\delta_s \in \{0,1\}$$
$$\lambda_s, \mu_s \geq 0, i = 1, \ldots, p$$



MILP: modeling tricks

A function must take a value out of N possible values

$$f(x) = b_1 \vee b_2 \vee \ldots \vee b_N$$

can be modeled as

$$f(x) = \sum_{i=1}^{N} b_i \delta_i$$

$$\sum_{i=1}^{N} \delta_i = 1$$

$$\delta_i \in \{0, 1\}, i = 1, \ldots, N$$

MILP: modeling tricks

Transforming integer variables into binary variables

Assume that

$$0 \leq x \leq u, z \in \mathbb{Z}.$$

If $2^N \leq u \leq 2^{N+1}$ then we can represent x using binary variables as follows:

$$x = \sum_{i=0}^{N} 2^i \delta_i, \quad \delta_i \in \{0, 1\}, i = 1 \ldots, N$$

MILP: modeling tricks

Linearizing the product of two binary variables

Let $y_1, y_2 \in \{0, 1\}$ two binary variables, and assume that its product, $y_1 y_2$, which is a nonlinear expression, appears in a given formulation. We can linearize the product as follows:

$$\delta \leq y_1$$
$$\delta \leq y_2$$
$$\delta \geq y_1 + y_2 - 1$$
$$\delta \in \{0, 1\}$$

Notice that $\delta = y_1 y_2$.

MILP: modeling tricks

Linearizing the product of a binary and a continous variable

Let z be a continuous variable such that $L \leq z \leq U$, and $x \in \{0, 1\}$ be a binary variable. Assume that its product, zx, which is a nonlinear expression, appears in a given formulation. We can linearize the product as follows:

$$\begin{aligned} y &\leq Ux \\ y &\geq Lx \\ z - y &\leq U(1 - x) \\ z - y &\geq L(1 - x) \end{aligned}$$

Notice that $y = zx$

MILP: modeling tricks

Example: A discrete competitive location problem under the probabilistic choice rule.

- A chain wants to enter in a given area by opening p facilities.
- Those facilities are to be open in p of the s potential sites pre-selected by the chain.
- There already exists m competing facilities operating in the area.
- Customers follow a probabilistic choice rule (they patronize all the facilities, and the amount spent at each facility is proportional to its attraction).
- The objective is to maximize the market share captured by the locating chain.

MILP: modeling tricks

Example: A discrete competitive location problem under the probabilistic choice rule.

Indices

i index for demand points (or customers), $i = \{1, \ldots, n\}$.

j index for the facilities,

$j = 1, \ldots, s$, for the potential new facilities,

$j = s + 1, \ldots, s + m$, for the existing competing facilities.

Data

w_i demand (or buying power) of demand point i.

d_{ij} distance between demand point i and location j.

a_{ij} quality of facility j as perceived by deman point i.

β modulator of the distance

MILP: modeling tricks

Example: A discrete competitive location problem under the probabilistic choice rule.

Computed data

$$u_{ij} = \frac{a_{ij}}{(d_{ij}+1)^{\beta}}$$ attraction that demand point i feels towards facility j.

Variables

$$x_j = \begin{cases} 1 & \text{if a facility is open at } j \\ 0 & \text{otherwise} \end{cases}, j = 1\ldots, s$$

MILP: modeling tricks

Example: A discrete competitive location problem under the probabilistic choice rule.

$$\max \quad \sum_{i=1}^{n} w_i \frac{\sum_{j=1}^{s} u_{ij} x_j}{\sum_{j=1}^{s} u_{ij} x_j + \sum_{j=s+1}^{s+m} u_{ij}}$$

$$\text{s.t.} \quad \sum_{j=1}^{s} x_j = p$$

$$x_j \in \{0, 1\}, j = 1, \ldots, s$$

MILP: modeling tricks

Example: A discrete competitive location problem under the probabilistic choice rule.

$$
\begin{aligned}
\max \quad & \sum_{i=1}^{n} w_i \frac{\sum_{j=1}^{s} u_{ij} x_j}{\sum_{j=1}^{s} u_{ij} x_j + \sum_{j=s+1}^{s+m} u_{ij}} \\
\text{s.t.} \quad & \sum_{j=1}^{s} x_j = p \\
& x_j \in \{0,1\}, j = 1, \ldots, s
\end{aligned}
$$

If we denote

$$
z_i = \frac{1}{\sum_{j=1}^{s} u_{ij} x_j + \sum_{j=s+1}^{s+m} u_{ij}}, i = 1, \ldots, n
$$

then the problem becomes

MILP: modeling tricks

Example: A discrete competitive location problem under the probabilistic choice rule.

$$
\begin{aligned}
\max \quad & \sum_{i=1}^{n} w_i z_i \sum_{j=1}^{s} u_{ij} x_j \\
\text{s.t.} \quad & z_i = \frac{1}{\sum_{j=1}^{s} u_{ij} x_j + \sum_{j=s+1}^{s+m} u_{ij}}, i = 1, \ldots, n \\
& \sum_{j=1}^{s} x_j = p \\
& x_j \in \{0, 1\}, j = 1, \ldots, s \\
& z_i \geq 0, i = 1, \ldots, n
\end{aligned}
$$

MILP: modeling tricks

Example: A discrete competitive location problem under the probabilistic choice rule.

$$
\begin{aligned}
\max \quad & \sum_{i=1}^{n} \sum_{j=1}^{s} w_i z_i u_{ij} x_j \\
\text{s.t.} \quad & z_i = \frac{1}{\sum_{j=1}^{s} u_{ij} x_j + \sum_{j=s+1}^{s+m} u_{ij}}, i = 1, \ldots, n \\
& \sum_{j=1}^{s} x_j = p \\
& x_j \in \{0, 1\}, j = 1, \ldots, s \\
& z_i \geq 0, i = 1, \ldots, n
\end{aligned}
$$

MILP: modeling tricks

Example: A discrete competitive location problem under the probabilistic choice rule.

$$
\begin{aligned}
\max \quad & \sum_{i=1}^{n} \sum_{j=1}^{s} (w_i z_i u_{ij}) x_j \\
\text{s.t.} \quad & z_i = \frac{1}{\displaystyle\sum_{j=1}^{s} u_{ij} x_j + \sum_{j=s+1}^{s+m} u_{ij}}, i = 1, \ldots, n \\
& \sum_{j=1}^{s} x_j = p \\
& x_j \in \{0, 1\}, j = 1, \ldots, s \\
& z_i \geq 0, i = 1, \ldots, n
\end{aligned}
$$

MILP: modeling tricks

Example: A discrete competitive location problem under the probabilistic choice rule.

If we denote

$$y_{ij} = (w_i z_i u_{ij})x_j, i = 1, \ldots, n, j = 1, \ldots, s$$

and taking into account that the product $y = zx$, where $L \leq z \leq U$ is continuous and x binary can be linearized as

$$\begin{aligned} y &\leq Ux \\ y &\geq Lx \\ z - y &\leq U(1-x) \\ z - y &\geq L(1-x) \end{aligned}$$

we have that the product $y_{ij} = (w_i z_i u_{ij})x_j$ can be linearized as follows

$$\left.\begin{array}{l} y_{ij} \leq w_i x_j, \\ y_{ij} \geq 0 x_j \Leftrightarrow y_{ij} \geq 0, \\ w_i z_i u_{ij} - y_{ij} \leq w_i(1-x_j), \\ w_i z_i u_{ij} - y_{ij} \geq 0(1-x_j) \Leftrightarrow w_i z_i u_{ij} - y_{ij} \geq 0, \end{array}\right\} i = 1, \ldots, n, j = 1, \ldots, s$$

MILP: modeling tricks

Example: A discrete competitive location problem under the probabilistic choice rule.

$$
\begin{aligned}
\max \quad & \sum_{i=1}^{n} \sum_{j=1}^{s} y_{ij} \\
\text{s.t.} \quad & z_i = \frac{1}{\sum_{j=1}^{s} u_{ij} x_j + \sum_{j=s+1}^{s+m} u_{ij}}, && i = 1, \ldots, n \\
& y_{ij} \leq w_i x_j, && i = 1, \ldots, n, j = 1, \ldots, s \\
& y_{ij} \geq 0, && i = 1, \ldots, n, j = 1, \ldots, s \\
& w_i z_i u_{ij} - y_{ij} \leq w_i (1 - x_j), && i = 1, \ldots, n, j = 1, \ldots, s \\
& w_i z_i u_{ij} - y_{ij} \geq 0, && i = 1, \ldots, n, j = 1, \ldots, s \\
& \sum_{j=1}^{s} x_j = p \\
& x_j \in \{0, 1\}, && j = 1, \ldots, s \\
& z_i \geq 0, && i = 1, \ldots, n \\
& y_{ij} \geq 0, && i = 1, \ldots, n, j = 1, \ldots, s
\end{aligned}
$$

MILP: modeling tricks

Example: A discrete competitive location problem under the probabilistic choice rule.

$$
\begin{aligned}
\max \quad & \sum_{i=1}^{n}\sum_{j=1}^{s} y_{ij} & \\
\text{s.t.} \quad & z_i = \frac{1}{\sum_{j=1}^{s} u_{ij}x_j + \sum_{j=s+1}^{s+m} u_{ij}}, & i = 1,\ldots,n \\
& y_{ij} \le w_i x_j, & i = 1,\ldots,n, j = 1,\ldots,s \\
& y_{ij} \ge 0, & i = 1,\ldots,n, j = 1,\ldots,s \\
& w_i z_i u_{ij} - y_{ij} \le w_i(1 - x_j), & i = 1,\ldots,n, j = 1,\ldots,s \\
& w_i z_i u_{ij} - y_{ij} \ge 0, & i = 1,\ldots,n, j = 1,\ldots,s \\
& \sum_{j=1}^{s} x_j = p & \\
& x_j \in \{0,1\}, & j = 1,\ldots,s \\
& z_i \ge 0, & i = 1,\ldots,n \\
& y_{ij} \ge 0, & i = 1,\ldots,n, j = 1,\ldots,s
\end{aligned}
$$

MILP: modeling tricks

Example: A discrete competitive location problem under the probabilistic choice rule.

$$z_i = \frac{1}{\sum_{j=1}^{s} u_{ij} x_j + \sum_{j=s+1}^{s+m} u_{ij}}$$

MILP: modeling tricks

Example: A discrete competitive location problem under the probabilistic choice rule.

$$z_i = \frac{1}{\sum_{j=1}^{s} u_{ij} x_j + \sum_{j=s+1}^{s+m} u_{ij}} \Leftrightarrow$$

$$z_i \left(\sum_{j=1}^{s} u_{ij} x_j + \sum_{j=s+1}^{s+m} u_{ij}\right) = 1$$

MILP: modeling tricks

Example: A discrete competitive location problem under the probabilistic choice rule.

$$z_i = \frac{1}{\sum_{j=1}^{s} u_{ij}x_j + \sum_{j=s+1}^{s+m} u_{ij}} \Leftrightarrow$$

$$z_i(\sum_{j=1}^{s} u_{ij}x_j + \sum_{j=s+1}^{s+m} u_{ij}) = 1 \Leftrightarrow$$

$$z_i\sum_{j=1}^{s} u_{ij}x_j + z_i\sum_{j=s+1}^{s+m} u_{ij} = 1$$

MILP: modeling tricks

Example: A discrete competitive location problem under the probabilistic choice rule.

$$z_i = \frac{1}{\sum_{j=1}^{s} u_{ij} x_j + \sum_{j=s+1}^{s+m} u_{ij}} \Leftrightarrow$$

$$z_i\left(\sum_{j=1}^{s} u_{ij} x_j + \sum_{j=s+1}^{s+m} u_{ij}\right) = 1 \Leftrightarrow$$

$$z_i \sum_{j=1}^{s} u_{ij} x_j + z_i \sum_{j=s+1}^{s+m} u_{ij} = 1 \Leftrightarrow$$

$$w_i z_i \sum_{j=1}^{s} u_{ij} x_j + w_i z_i \sum_{j=s+1}^{s+m} u_{ij} = w_i$$

MILP: modeling tricks

Example: A discrete competitive location problem under the probabilistic choice rule.

$$z_i = \frac{1}{\sum_{j=1}^{s} u_{ij} x_j + \sum_{j=s+1}^{s+m} u_{ij}} \Leftrightarrow$$

$$z_i \left(\sum_{j=1}^{s} u_{ij} x_j + \sum_{j=s+1}^{s+m} u_{ij}\right) = 1 \Leftrightarrow$$

$$z_i \sum_{j=1}^{s} u_{ij} x_j + z_i \sum_{j=s+1}^{s+m} u_{ij} = 1 \Leftrightarrow$$

$$w_i z_i \sum_{j=1}^{s} u_{ij} x_j + w_i z_i \sum_{j=s+1}^{s+m} u_{ij} = w_i \Leftrightarrow$$

$$\sum_{j=1}^{s} w_i z_i u_{ij} x_j + w_i z_i \sum_{j=s+1}^{s+m} u_{ij} = w_i$$

MILP: modeling tricks

Example: A discrete competitive location problem under the probabilistic choice rule.

$$z_i = \frac{1}{\sum_{j=1}^{s} u_{ij} x_j + \sum_{j=s+1}^{s+m} u_{ij}} \Leftrightarrow$$

$$z_i \left(\sum_{j=1}^{s} u_{ij} x_j + \sum_{j=s+1}^{s+m} u_{ij}\right) = 1 \Leftrightarrow$$

$$z_i \sum_{j=1}^{s} u_{ij} x_j + z_i \sum_{j=s+1}^{s+m} u_{ij} = 1 \Leftrightarrow$$

$$w_i z_i \sum_{j=1}^{s} u_{ij} x_j + w_i z_i \sum_{j=s+1}^{s+m} u_{ij} = w_i \Leftrightarrow$$

$$\sum_{j=1}^{s} w_i z_i u_{ij} x_j + w_i z_i \sum_{j=s+1}^{s+m} u_{ij} = w_i \Leftrightarrow$$

$$\sum_{j=1}^{s} y_{ij} + w_i z_i \sum_{j=s+1}^{s+m} u_{ij} = w_i$$

$$
\begin{array}{lll}
\max & \displaystyle\sum_{i=1}^{n}\sum_{j=1}^{s} y_{ij} & \\
\text{s.t.} & \sum_{j=1}^{s} y_{ij} + w_i z_i \sum_{j=s+1}^{s+m} u_{ij} = w_i, & i = 1,\dots,n \\
 & y_{ij} \le w_i x_j, & i = 1,\dots,n, j = 1,\dots,s \\
 & y_{ij} \ge 0, & i = 1,\dots,n, j = 1,\dots,s \\
 & w_i z_i u_{ij} - y_{ij} \le w_i(1 - x_j), & i = 1,\dots,n, j = 1,\dots,s \\
 & w_i z_i u_{ij} - y_{ij} \ge 0 & i = 1,\dots,n, j = 1,\dots,s \\
 & \displaystyle\sum_{j=1}^{s} x_j = p & \\
 & x_j \in \{0,1\}, & j = 1,\dots,s \\
 & z_i \ge 0, & i = 1,\dots,n \\
 & y_{ij} \ge 0, & i = 1,\dots,n, j = 1,\dots,s
\end{array}
$$

MILP: modeling tricks

Example: A discrete competitive location problem under the probabilistic choice rule.

$$
\begin{array}{lll}
\max & \displaystyle\sum_{i=1}^{n}\sum_{j=1}^{s} y_{ij} & \\
\text{s.t.} & \sum_{j=1}^{s} y_{ij} + w_i z_i \sum_{j=s+1}^{s+m} u_{ij} \leq w_i, & i = 1,\ldots,n \\
& y_{ij} \leq w_i x_j, & i = 1,\ldots,n, j = 1,\ldots,s \\
& y_{ij} \geq 0, & i = 1,\ldots,n, j = 1,\ldots,s \\
& w_i z_i u_{ij} - y_{ij} \leq w_i(1 - x_j), & i = 1,\ldots,n, j = 1,\ldots,s \\
& w_i z_i u_{ij} - y_{ij} \geq 0 & i = 1,\ldots,n, j = 1,\ldots,s \\
& \displaystyle\sum_{j=1}^{s} x_j = p & \\
& x_j \in \{0,1\}, & j = 1,\ldots,s \\
& z_i \geq 0, & i = 1,\ldots,n \\
& y_{ij} \geq 0, & i = 1,\ldots,n, j = 1,\ldots,s
\end{array}
$$

MILP: modeling tricks

Let x be a continuous variable such that $L \leq x \leq U$. And let $\delta \in \{0, 1\}$ be a binary variable.

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Let x be a continuous variable such that $L \leq x \leq U$. And let $\delta \in \{0, 1\}$ be a binary variable.

Conditional constraints 2

$$\delta = 0 \Longrightarrow x \leq 0$$

can be modeled as

$$x \leq \delta U.$$

Since $P \Rightarrow Q$ is equivalent to $\neg Q \Rightarrow \neg P$ the previous expression also models

$$x > 0 \Longrightarrow \delta = 1$$

MILP: modeling tricks

Let x be a continuous variable such that $L \leq x \leq U$. And let $\delta \in \{0, 1\}$ be a binary variable.

Conditional constraints 3

$$\delta = 0 \Longrightarrow x \geq 0$$

can be modeled as

$$x \geq \delta L.$$

Since $P \Rightarrow Q$ is equivalent to $\neg Q \Rightarrow \neg P$ the previous expression also models

$$x < 0 \Longrightarrow \delta = 1$$

MILP: modeling tricks

Let $\epsilon > 0$ be a small number, and m and M two constants such that $m \leq f(x) - b \leq M$ for any feasible value of x. And let $\delta \in \{0, 1\}$ be a binary variable.

MILP: modeling tricks

Let $\epsilon > 0$ be a small number, and m and M two constants such that $m \leq f(x) - b \leq M$ for any feasible value of x. And let $\delta \in \{0, 1\}$ be a binary variable.

Conditional constraints 4 (type $\leq$)

$$\delta = 1 \Longrightarrow f(x) \leq b$$

can be modeled as

$$f(x) \leq b + M(1 - \delta).$$

Since $P \Rightarrow Q$ is equivalent to $\neg Q \Rightarrow \neg P$ the previous expression also models

$$f(x) > b \Longrightarrow \delta = 0$$

MILP: modeling tricks

Let $\epsilon > 0$ be a small number, and m and M two constants such that $m \leq f(x) - b \leq M$ for any feasible value of x. And let $\delta \in \{0, 1\}$ be a binary variable.

Conditional constraints 5 (type $\leq$)

$$f(x) \leq b \Longrightarrow \delta = 1$$

is equivalent to

$$\delta = 0 \Longrightarrow f(x) > b$$

which can be tranformed into

$$\delta = 0 \Longrightarrow f(x) \geq b + \epsilon.$$

The previous expressions can be both modeled as

$$f(x) \geq b + \epsilon + (m - \epsilon)\delta$$

MILP: modeling tricks

Let $\epsilon > 0$ be a small number, and m and M two constants such that $m \leq f(x) - b \leq M$ for any feasible value of x. And let $\delta \in \{0, 1\}$ be a binary variable.

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Let $\epsilon > 0$ be a small number, and m and M two constants such that $m \leq f(x) - b \leq M$ for any feasible value of x. And let $\delta \in \{0, 1\}$ be a binary variable.

Conditional constraints 6 (type $\geq$)

$$\delta = 1 \Longrightarrow f(x) \geq b$$

can be modeled as

$$f(x) \geq b + m(1 - \delta).$$

Since $P \Rightarrow Q$ is equivalent to $\neg Q \Rightarrow \neg P$ the previous expression also models

$$f(x) < b \Longrightarrow \delta = 0$$

MILP: modeling tricks

Let $\epsilon > 0$ be a small number, and m and M two constants such that $m \leq f(x) - b \leq M$ for any feasible value of x. And let $\delta \in \{0, 1\}$ be a binary variable.

Conditional constraints 7 (type $\geq$)

$$f(x) \geq b \Longrightarrow \delta = 1$$

is equivalent to

$$\delta = 0 \Longrightarrow f(x) < b$$

which can be transformed into

$$\delta = 0 \Longrightarrow f(x) \leq b - \epsilon.$$

The previous expressions can be both modeled as

$$f(x) \leq b - \epsilon + (M + \epsilon)\delta$$

MILP: modeling tricks

Let $\epsilon > 0$ be a small number, and m and M two constants such that $m \leq f(x) - b \leq M$ for any feasible value of x. And let $\delta \in \{0, 1\}$ be a binary variable.

MILP: modeling tricks

Let $\epsilon > 0$ be a small number, and m and M two constants such that $m \leq f(x) - b \leq M$ for any feasible value of x. And let $\delta \in \{0, 1\}$ be a binary variable.

Conditional constraints 8 (type $=$)

$$\delta = 1 \Longrightarrow f(x) = b \text{ is equivalent to } \delta = 1 \Longrightarrow \begin{cases} f(x) \leq b \\ f(x) \geq b \end{cases}$$

Hence, it can be modeled by the constraints

$$\begin{aligned} f(x) &\leq b + M(1 - \delta) \\ f(x) &\geq b + m(1 - \delta) \end{aligned}$$

Since $P \Rightarrow Q$ is equivalent to $\neg Q \Rightarrow \neg P$ the previous expression also models

$$f(x) \neq b \Longrightarrow \delta = 0$$

MILP: modeling tricks

Let $\epsilon > 0$ be a small number, and m and M two constants such that $m \leq f(x) - b \leq M$ for any feasible value of x. And let $\delta \in \{0, 1\}$ be a binary variable.

Conditional constraints 9 (type $=$)

$f(x) = b \Longrightarrow \delta = 1$ is equivalent to

$$\begin{gathered} f(x) \leq b \Longrightarrow \delta_1 = 1 \\ f(x) \geq b \Longrightarrow \delta_2 = 1 \\ \left.\begin{array}{l} \delta_1 = 1 \\ \delta_2 = 1 \end{array}\right\} \Longrightarrow \delta = 1 \\ \delta_1, \delta_2 \in \{0, 1\} \end{gathered}$$

which can be modeled as

$$\begin{gathered} f(x) \geq b + \epsilon + (m - \epsilon)\delta_1 \\ f(x) \leq b - \epsilon + (M + \epsilon)\delta_2 \\ \delta_1 + \delta_2 - \delta \leq 1 \\ \delta_1, \delta_2 \in \{0, 1\} \end{gathered}$$

MILP: modeling tricks

Let $\epsilon > 0$ be a small number, and m and M two constants such that $m \leq f(x) - b \leq M$ for any feasible value of x. And let $\delta \in \{0, 1\}$ be a binary variable.

Conditional constraints 9 (type $=$)

Since $f(x) = b \Longrightarrow \delta = 1$ is equivalent to $\delta = 0 \Longrightarrow f(x) \neq b$

this last conditional constraint can also be modeled as

$$f(x) \geq b + \epsilon + (m - \epsilon)\delta_1$$
$$f(x) \leq b - \epsilon + (M + \epsilon)\delta_2$$
$$\delta_1 + \delta_2 - \delta \leq 1$$
$$\delta_1, \delta_2 \in \{0, 1\}$$

MILP: modeling tricks

Let $\epsilon > 0$ be a small number, and m and M two constants such that $m \leq f(x) - b \leq M$ for any feasible value of x. And let $\delta \in \{0, 1\}$ be a binary variable.

Conditional constraints 10: double implications

Double implications can be transformed into two unidirectional implications. For instance

$$\delta = 1 \iff f(x) \leq b$$

is equivalent to

$$\begin{cases} \delta = 1 & \implies & f(x) \leq b \\ f(x) \leq b & \implies & \delta = 1 \end{cases}$$

Hence, it can be modeled as

$$f(x) \leq b + M(1 - \delta)$$
$$f(x) \geq b + \epsilon + (m - \epsilon)\delta$$

MILP: modeling tricks

Let $\epsilon > 0$ be a small number, and m and M two constants such that $m \leq f(x) - b \leq M$ for any feasible value of x. And let $\delta \in \{0,1\}$ be a binary variable.

Conditional constraints 10: double implications

$$\delta = 1 \Longleftrightarrow f(x) \geq b$$

can be modeled as

$$f(x) \geq b + m(1-\delta)$$
$$f(x) \leq b - \epsilon + (M+\epsilon)\delta$$

MILP: modeling tricks

Let $\epsilon > 0$ be a small number, and m and M two constants such that $m \leq f(x) - b \leq M$ for any feasible value of x. And let $\delta \in \{0, 1\}$ be a binary variable.

Conditional constraints 10: double implications

$$\delta = 1 \Longleftrightarrow f(x) = b$$

can be modeled as

$$\begin{gathered} f(x) \leq b + M(1 - \delta) \\ f(x) \geq b + m(1 - \delta) \\ f(x) \geq b + \epsilon + (m - \epsilon)\delta_1 \\ f(x) \leq b - \epsilon + (M + \epsilon)\delta_2 \\ \delta_1 + \delta_2 - \delta \leq 1 \\ \delta_1, \delta_2 \in \{0, 1\} \end{gathered}$$

MILP: modeling tricks

Equivalences for conditional propositions

The following equivalences can be used before converting them into constraints:

$P \Rightarrow Q$	$\neg P \vee Q$
$P \Rightarrow (Q \wedge R)$	$(P \Rightarrow Q) \wedge (P \Rightarrow R)$
$P \Rightarrow (Q \vee R)$	$(P \Rightarrow Q) \vee (P \Rightarrow R)$
$(P \wedge Q) \Rightarrow R$	$(P \Rightarrow R) \vee (Q \Rightarrow R)$
$(P \vee Q) \Rightarrow R$	$(P \Rightarrow R) \wedge (Q \Rightarrow R)$
$\neg(P \vee Q)$	$(\neg P) \wedge (\neg Q)$
$\neg(P \wedge Q)$	$(\neg P) \vee (\neg Q)$

MILP: modeling tricks

Assume that the indicator variable δ_i is equal to 1 when the constraint C_i holds:

$$\delta_i = \begin{cases} 1 & \text{if } C_i \text{ holds} \\ 0 & \text{otherwise} \end{cases}$$

Simple conditional or composed statements

$C_1 \vee C_2$	$\delta_1 + \delta_2 \geq 1$
$C_1 \wedge C_2$	$\delta_1 + \delta_2 = 2$
$\neg C_1$	$\delta_1 = 0$
$C_1 \implies C_2$	$\delta_1 \leq \delta_2$
$C_1 \iff C_2$	$\delta_1 = \delta_2$

MILP: modeling tricks

Complex conditional or composed statements

Complex conditional or composed statements are decomposed into two implications in order to model them easier.

Example

$$(C_1 \vee C_2) \Longrightarrow (C_3 \vee C_4 \vee C_5)$$

is modeled as

$$(\delta_1 + \delta_2 \geq 1) \Longrightarrow (\delta_3 + \delta_4 + \delta_5 \geq 1)$$

which, in turn, can be transformed into

$$(\delta_1 + \delta_2 \geq 1) \Rightarrow \delta = 1 \Rightarrow (\delta_3 + \delta_4 + \delta_5 \geq 1)$$

or more clearly,

$$\begin{cases} (\delta_1 + \delta_2 \geq 1) \Rightarrow \delta = 1 \\ \delta = 1 \Rightarrow (\delta_3 + \delta_4 + \delta_5 \geq 1) \end{cases}$$

which becomes

$$\begin{cases} \delta_1 + \delta_2 \leq 2\delta \\ \delta \leq \delta_3 + \delta_4 + \delta_5 \end{cases}$$

MILP: modeling tricks

Example

$$(x \leq b) \wedge (x \geq 1) \implies (y = z + 1)$$

is first transformed into

$$(x \leq b) \wedge (x \geq 1) \implies \delta = 1 \implies (y = z + 1)$$

and this in turn is written as

$$\begin{array}{lll}
(x \leq b) \Rightarrow \delta_1 = 1 & & x \geq b + \epsilon + (m_1 - \epsilon)\delta_1 \\
(x \geq 1) \Rightarrow \delta_2 = 1 & & x \leq 1 - \epsilon + (M_1 + \epsilon)\delta_2 \\
(\delta_1 = 1) \wedge (\delta_1 = 1) \Rightarrow \delta = 1 & \text{which becomes} & \delta_1 + \delta_2 - \delta \leq 1 \\
(\delta = 1) \Rightarrow (y \geq z + 1) & & y - z \geq 1 + m_2(1 - \delta) \\
(\delta = 1) \Rightarrow (y \leq z + 1) & & y - z \leq 1 + M_2(1 - \delta)
\end{array}$$

where $\epsilon > 0$ is a small number and $m_1 \leq x - b$, $M_1 \geq x - 1$, $m_2 \leq y - z - 1 \leq M_2$.

MILP: modeling tricks

More tricks have been designed to:

- Define nonconvex polygonal regions throught a set of constraints.
- Work with Special Ordered Sets of type 1 (SOS1), where in a set of variables only one of them can have a value different from 0, and SOS2, where in a set a variables at most two of them can be different from 0 and they must be consecutive variables.

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Sometimes the same problem can be modeled in different ways. A formulation ‘A’ is said to be better (stronger) than another formulation ‘B’ if the feasible set of ‘A’ is included in the feasible set of ‘B’.

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Interestingly, in MILP sometimes it is better a formulation with a bigger number of variables and constraints!