Empirical investigation of the convergence speed of inclusion functions in a global optimization context *

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Abstract. This paper deals with the empirical convergence speed of inclusion functions applied in interval methods for global optimization. According to our experience the natural interval extension of a given function can be as good as a usual quadratically convergent inclusion function, and although centered forms are in general only of second-order, they can perform as one of larger convergence order. These facts indicate that the theoretical convergence order should not be the only indicator of the quality of an inclusion function, it would be better to know which inclusion function can be used most efficiently in concrete instances. For this reason we have investigated the empirical convergence speed of the usual inclusion functions on some test functions.

Keywords: interval arithmetic, global optimization, inclusion function, convergence order, empirical convergence speed.

AMS Subject Classification: 65K05, 90C30, 65G30, 65G40

1. Introduction

The aim of this paper is to analyze the convergence speed of inclusion functions used in branch and bound algorithms solving global optimization problems [1]. In such methods inclusion functions play an important role providing bounds on the range of the objective functions [9, 18].

The comparison of inclusion functions is usually done by comparing their convergence orders. The inclusion function with the higher convergence order is deemed to be the better function. However, the convergence order only provides a lower bound for the speed, i.e. it shows the worst case. For practical purposes one would like to know the average behaviour, especially for actually used, finite length argument interval sequences. We have therefore made an extensive empirical

* This work has been supported by the Grants OTKA T 034350 and T 032118, OMFB D-30/2000, and OMFB E-24/2001. The authors are grateful for the anonymous referees for their suggestions.

investigation measuring the average speed so that we can decide which inclusion function should be used in a given situation.

The context of our study is unconstrained global optimization, thus the general problem can be defined as

$$\min_{x \in X} f(x),$$

where f is a twice continuously differentiable function defined over the n-dimensional interval $X \subset \mathbb{R}^n$.

Real numbers are denoted by $x, y, \ldots$ and compact intervals by $X = [\underline{X}, \overline{X}], Y = [\underline{Y}, \overline{Y}], \ldots$, where $\underline{X} = \min\{x \in X\}$ and $\overline{X} = \max\{x \in X\}$ are the lower and upper bounds of X, respectively. The set of compact intervals is denoted by $\mathbb{I} := \{[a, b] \mid a, b \in \mathbb{R},\ a \leq b\}$.

The notation $x = (x_1, \ldots, x_n)^T$ and $X = (X_1, \ldots, X_n)^T$, where $x_i \in \mathbb{R},\ X_i \in \mathbb{I}\ (i = 1, \ldots, n)$ is used for real and interval vectors. The set of n-dimensional intervals (also called boxes) is denoted by $\mathbb{I}^n$.

The width of the interval X is defined by $\mathrm{w}(X) = \overline{X} - \underline{X}$, if $X \in \mathbb{I}$, and $\mathrm{w}(X) = \max_{i=1,\ldots,n} \mathrm{w}(X_i)$, when $X \in \mathbb{I}^n$. The midpoint is defined by $\mathrm{mid}(X) = (\underline{X} + \overline{X})/2$, where $\underline{X} = (\underline{X}_1, \ldots, \underline{X}_n)$, $\overline{X} = (\overline{X}_1, \ldots, \overline{X}_n)$. The relative width is defined by $w_{rel}(X) = \mathrm{w}(X)/\max\{1, \min_{x \in X} |x|\}$. For $X \in \mathbb{I}^n$ $w_{rel}(X) = \max_{i=1,\ldots,n} w_{rel}(X_i)$.

Let $f : Y \to \mathbb{R}$ $(Y \in \mathbb{I}^n)$ be a continuous function, and $\mathbb{I}^n(Y) = \{X \mid X \in \mathbb{I}^n, X \subseteq Y\}$.

DEFINITION 1. *The function $F : \mathbb{I}^n(Y) \to \mathbb{I}$ is an* inclusion function *of f, if for every $X \in \mathbb{I}^n(Y)$ and $x \in X$, $f(x) \in F(X)$, i.e. the range $f(X)$ is in $F(X)$.*

An inclusion function generally overestimates the range of a function. The extent of the overestimation depends on the inclusion function type, on the function to be included and on the width of the interval. Of course the better the inclusion function, the smaller the overestimation (if the computational costs are the same).

DEFINITION 2. *An inclusion function F is called* inclusion isotone, *if $F(X) \subseteq F(Z)$ holds for every $X \subseteq Z$.*

An inclusion function F is called *inclusion isotone from above (below)*, if $\overline{F}(X) \leq \overline{F}(Z)$ ($\underline{F}(X) \geq \underline{F}(Z)$) holds for every $X \subseteq Z$.

DEFINITION 3. *An inclusion function F of f is called* zero-convergent, *if* $\mathrm{w}(F(X)) - \mathrm{w}(f(X)) \to 0$ *while* $\mathrm{w}(X) \to 0$.

The most studied property for an inclusion function is the convergence order, what is defined as follows.

DEFINITION 4. The convergence order *of an inclusion function F of f is $\alpha(>0)$, if the inequality*

$$\mathrm{w}(F(X)) - \mathrm{w}(f(X)) \leq c\mathrm{w}(X)^{\alpha} \tag{1}$$

holds for every $X \subseteq Y$ for a real c.

From this definition it is easy to see, that every inclusion function is zero-convergent if it is α-convergent with $\alpha > 0$. In general an inclusion function with larger convergence order is better, although it may also happen that an inclusion function with larger α has bigger overestimation for some intervals due to the constant c.

2. The empirical convergence speed

The convergence order of an inclusion function can usually be determined theoretically. It gives a lower bound for the convergence order, due to of the worst case definition. Our goal is to determine an approximate speed which gives a good indication of which inclusion function should be used in order to have the best inclusion at an acceptable cost in a given situation.

The empirical convergence speed can be obtained by determining the constants α and c approximating the equation

$$\mathrm{w}(F(X)) - \mathrm{w}(f(X)) = c \cdot \mathrm{w}(X)^{\alpha} \tag{2}$$

for a finite set of inclusions with the smallest error. This fits to the definition of the theoretical convergence order (1), except that the inequality is changed to equality in order to obtain information about the average behaviour on a finite set of inclusions. Changing the examined set of inclusions a lot of information is lost. If we consider that we are interested in the convergence speed of inclusion functions in global optimization, and the set of all inclusions extracted by a global optimization method is examined, the result is satisfactory.

Reformulating equality (2) to

$$\lg\Big(\mathrm{w}(F(X)) - \mathrm{w}(f(X))\Big) = \lg(c) + \alpha \lg\Big(\mathrm{w}(X)\Big),$$

we can determine the α and c constants easily with linear regression. This realigning is possible, because the overestimation can never be negative. Clearly, the estimation of the parameters with this new equality is not equivalent to the estimation with equality (2), but as we will see on the results, it's still adequate. It is obviously not the only way to extract suitable information from the actual overestimation figures, still the description parameters of α and c make it quite useful when estimating the actual convergence speed.

3. The examined sets of boxes

One of the main questions for the empirical convergence speed is how to choose the set of boxes that are applied in the computation. In our investigation the following sets were examined:

- a sequence of embedded boxes containing a given global minimizer point,
- several random sequences of embedded boxes converging to a non-stationary point,
- a sequence of boxes increasing in their width starting from the given search interval,
- boxes generated by the Moore-Skelboe optimization method.

The set obtained by the Moore-Skelboe optimization method is the most interesting one. The processed boxes depend on the inclusion functions, accelerating tools, branching and selecting rules used. A general algorithm with natural interval extension and centered form was used together with the monotonicity and cutoff tests as accelerating tools. The bisection by the widest interval component was used for branching, and we select the subinterval with the smallest lower bound for the next subdivision.

For every box of the sequences we compute all the studied inclusion functions, the interval width, and the approximation of the range. The hardest part of the computations is the range approximation, because it means that we have to approximate the solution of an NP-hard problem for every box. For this reason one possible way is to use the Moore-Skelboe algorithm itself to approximate the lower bound by minimizing the function, and to obtain the upper bound by maximizing it. For the optimization we used all the available accelerating tools within the Moore-Skelboe algorithm: the cutoff test, the monotonicity test, the concavity test, and the Newton-step. The applied stopping criterion required the relative width of the inclusion of the minimum (or maximum) small enough, the maximal relative error of the approximation must be below a small positive value (in the present case it was set to 10^{-8}).

4. The examined inclusion function types

In the literature of interval global optimization one can find various methods for constructing inclusion functions. Some of them can be used for every continuous functions, even for non-differentiable ones, while others better only for polynomial functions or only for one-dimensional functions [19]. In other words the used information can be very different from the simple methods that use only function information to very sophisticated ones that use higher derivative information [10]. Our aim is to analyze some inclusion functions, that can be applied for the general optimization test functions. For this investigation, we examined only the most widely used ones, which are available in many interval libraries.

4.1. Natural interval extension

This inclusion function was first studied in detail by Ramon E. Moore [12]. It is also called naive interval extension, since it expands the usual arithmetic operations in a simple, straightforward way from reals to intervals. The natural inclusion function of an implemented function which is given as a computer program, can be given by substituting the real variables for intervals and the real operations or standard functions for interval operations or the respective inclusion functions [7, 15].

Denote the natural inclusion function for a function f by $F_n(X)$. It was shown (see for example [14]), that the convergence order of the natural inclusion function is at least 1.

4.2. Centered form or mean value form

This inclusion function uses also the derivative of the function, thus we can apply this method only if the function is differentiable and we can compute the enclosure of the derivative. The derivative is usually easy to obtain by automatic differentiation. The centered form is usually defined as

$$F_c(X) = F(c) + F'(X)(X - c),$$

where $c = \mathrm{mid}(X)$. Although we use often the midpoint for the c parameter, still it is reasonable to choose it elsewhere inside X. In case the convergence order of the inclusion function $F'(X)$ of the derivative is at least one, then the convergence order of the centered form is at least quadratic [8].

4.3. Baumann's optimal centered form

Baumann introduced the optimal c for the centered form in [2]. He proved that $\underline{F_c}(X) \leq \underline{F_{b^-}}(X), \forall c \in X$ for an optimal $b^- \in X$. The optimal lower bound is attained at $c = b^-$, that can be obtained in the one dimensional case as

$$b^- = \frac{U\underline{X} - L\overline{X}}{U - L},$$

where $X = [\underline{X}, \overline{X}]$ and $F'(X) = [L, U] \ni 0$. In case $F'(X) > 0$, the optimal center is $b^- = \overline{X}$, and when $F'(X) < 0$ it is $b^- = \underline{X}$. For the optimal upper bound attained at b^+ one should reflect b^- on the midpoint of the interval X. The best inclusion can be obtained using both b^+ and b^-, as we did in our computations.

In higher dimensions the above formula must be used componentwise. Although the Baumann's form gives always at least as good enclosure as the general centered form, still its convergence order is only quadratic.

4.4. Slope arithmetic

Instead of the inclusion of the first derivative, the so called slope is usually better for building an enclosure of the underlying function [4, 16]. The slope itself is a lower and upper bounding line going through a given basis point within the given interval. The slope can be determined in a similar way as it is made in automatic differentiation, or in object oriented implementation of naive interval arithmetic: by operator overloading. For more details see the last cited volume.

5. Theoretical results

Let us consider that $F_i, (i = 1, \ldots, n)$ are different inclusion functions of the continuously differentiable function $f : Y \subseteq \mathbb{R}^n \to \mathbb{R}$. Let us define a new inclusion function F, as the intersection of F_i for all $i = 1, \ldots, n$, i.e. for every $X \subseteq Y$

$$F(X) = \bigcap_{i=1,\ldots,n} F_i(X).$$

Inclusion isotonicity has an important role in verified numerical methods, especially in global optimization techniques. In this context the following statements hold.

PROPOSITION 1. *Assume that* F_i, $i = 1, \ldots, n$ *are inclusion isotone. Then the inclusion function* $F = \bigcap_{i=1,\ldots,n} F_i(X)$ *is also inclusion isotone.*

Proof. We prove the statement indirectly. Assume that there exist intervals $X \subseteq Z$ such that $F(X) \nsubseteq F(Z)$. Due to the assumption of the proposition the inclusions $F_i(X) \subseteq F_i(Z)$, $i = 1, \ldots, n$ hold. Let us suppose that $\underline{F}(X) < \underline{F}(Z)$, i.e.

$$\max_{i=1,\ldots,n} \underline{F}_i(X) < \max_{i=1,\ldots,n} \underline{F}_i(Z). \tag{3}$$

By using $\underline{F}_i(X) \geq \underline{F}_i(Z)$, $i = 1, \ldots, n$ we obtain that

$$\max_{i=1,\ldots,n} \underline{F}_i(X) \geq \max_{i=1,\ldots,n} \underline{F}_i(Z),$$

and that contradicts (3). For the case $\overline{F}(X) > \overline{F}(Z)$ similar arguments hold.

PROPOSITION 2. *Assume that* $\exists\ i \in \{i_1, \ldots, i_m\} \subseteq \{1, \ldots, n\}$ *and* $\exists\ k \in \{k_1, \ldots, k_l\} \subseteq \{1, \ldots, n\}$ *such that* $\underline{F}_i(X) \geq \underline{F}_j(X)$ *and* $\overline{F}_k(X) \leq \overline{F}_j(X)$ *for every* $X \subseteq Y$, $j = 1, \ldots, n$, *while* F_i *and* F_k *are inclusion isotone from below and from above, respectively for all* $i \in \{i_1, \ldots, i_m\}$ *and* $k \in \{k_1, \ldots, k_l\}$. *Then the inclusion function* $F = \bigcap_{i=1,\ldots,n} F_i(X)$ *is also inclusion isotone.*

Proof. Due to the conditions $F(X) = [\underline{F}_i(X), \overline{F}_k(X)]$ $i \in \{i_1, \ldots, i_m\}$ and $k \in \{k_1, \ldots, k_l\}$ for every $X \subseteq Y$, F combines the one-sided inclusion isotonicity of F_i-s and F_k-s.

In branch and bound optimization methods the inclusion isotonicity can be reached easily when for all the descents X_k of X the inclusion function is intersected with $F(X)$. It would be interesting to see how much the improvement of this easy trick can be for a non-isotone inclusion function, e.g. for the general centered form.

PROPOSITION 3. *Suppose that the convergence orders of* F_i *are* α_i, $i = 1, \ldots, n$, *respectively, the convergence order of the inclusion function* $F = \bigcap_{i=1,\ldots,n} F_i(X)$ *is at least* $\max_{i=1,\ldots,n} \alpha_i$.

Proof. By the construction of F it is trivial that $F(X) \subseteq F_i(X)$, $\forall X \subseteq Y$, $i = 1, \ldots, n$, thus $\mathrm{w}(F(X)) \leq \mathrm{w}(F_i(X))$, $i = 1, \ldots, n$ and

$$\mathrm{w}(F(X)) - \mathrm{w}(f(X)) \leq \mathrm{w}(F_i(X)) - \mathrm{w}(f(X)) \leq c_i \mathrm{w}(X)^{\alpha_i},$$

for all $i = 1, \ldots, n$ (where c_i is the respective constant valid for the convergence speed of the inclusion function F_i), and that proves our statement.

And now let us consider the zero convergence property for the intersection of inclusion functions.

COROLLARY 1. *For the zero-convergence of F it is sufficient to have at least one $i \in \{1, \ldots, n\}$ such that F_i is α-convergent (with $\alpha_i > 0$).*

6. Numerical results

The most interesting results of our experiments are shown in the following figure and tables. For the computation the C-XSC-2.0 interval library was used with the CToolbox-2.0 software that contained the optimization code [6]. A graphical representation of the results for the Goldstein-Price function can be seen in Figure 1. The six graphs can be understood in the following way.

The first five graphs belong to the five examined inclusion functions, while the last one shows a summary. In every graph the horizontal and vertical axes show the logarithm of the width of the box and the logarithm of the overestimation, respectively. The small black circles denote the overestimation of intervals. Sometimes the inclusion functions give the exact range — at least according to our estimated range. In these cases the logarithm of the overestimation is not defined. It should be $-\infty$, but for a reasonable result we adjust the logarithm of the overestimation to -30 (it reflects in a way the limitation of computer number representation). Every graph provides a line, what we have obtained by linear regression. From the legend one can read the approximate convergence order and the order of magnitude of the c constant. On the upper part of every graph we denote with small triangles when the given inclusion function gave the best lower or upper bound.

Note, that in the case of centered and Baumann inclusion functions for small intervals (around a diameter of 10^{-10}) the overestimation is closely the same. It is why we have obtained an almost constant line there. This happens because we reach the level of rounding error, what means that the inclusion differs from the range approximation only by rounding error.

Comparing the results of the centered form and Baumann's optimal centered form note, that theoretically the Baumann's form is at least as good, but in some cases the empirical convergence speed of the plain centered form is slightly better in the parameter α (see the results for the function of Goldstein and Price on Figure 1). This can happen only since the constant c is smaller in these cases for the optimal center.

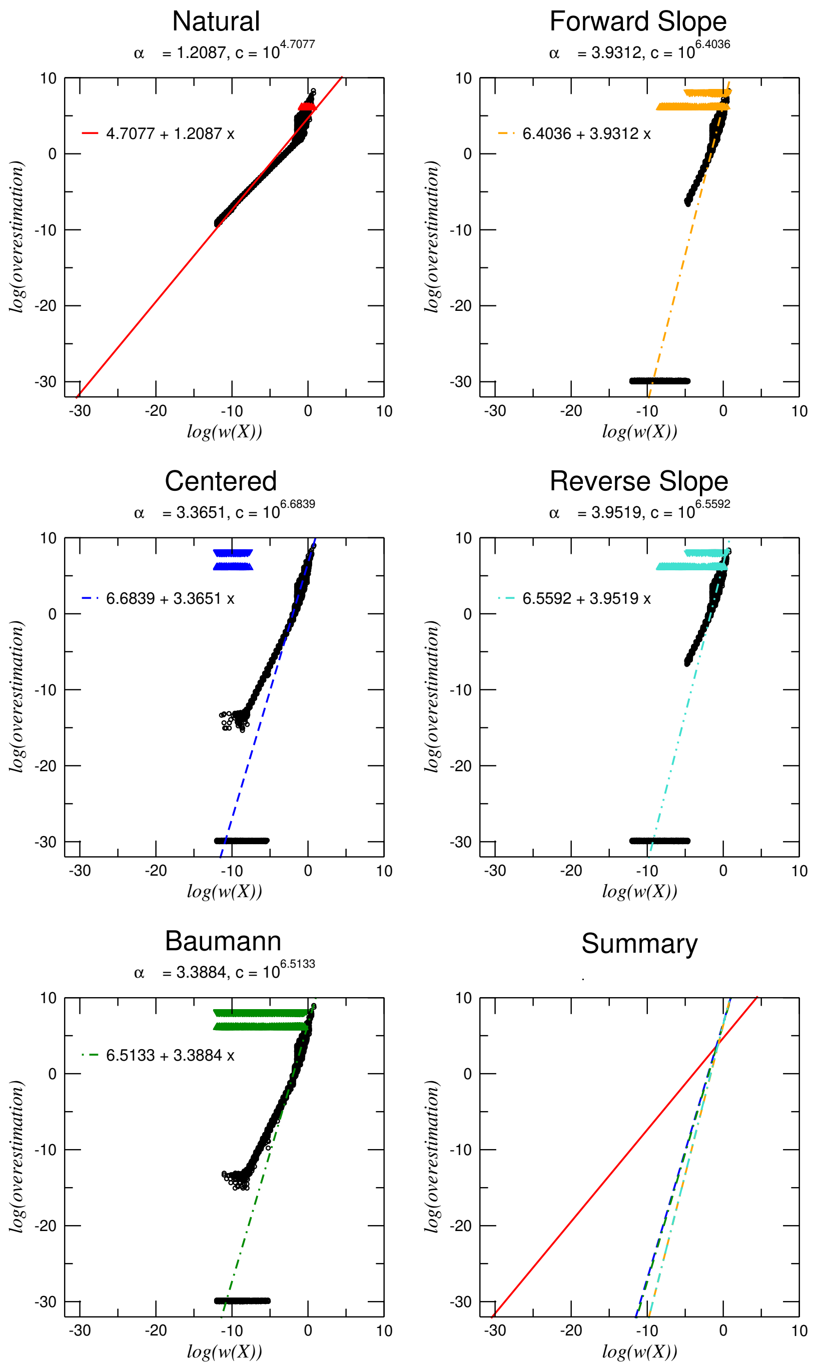

Figure 1. Empirical convergence speed results on the Goldstein-Price function for boxes extracted by the Moore-Skelboe B&B algorithm.

Evaluating the consequences of the results on Figure 1, we can conclude that as the last graph indicates, the natural interval extension is the best choice if our interval is relatively large, ca. of width 0.01 and wider. Otherwise the forward or reverse mode slope arithmetics [4, 16] provide the best inclusion. These findings reflect only the case when the subintervals are produced by a branch-and-bound algorithm searching for the global minimum — i.e. for other interval sequences different conclusions can be drawn.

Otherwise the figure shows nicely that the natural interval extension provides a close to linear convergence also according to our empirical convergence study.

The more sophisticated techniques that apply the inclusion of the first derivative achieve substantially better convergence speed: the slope arithmetics around $\alpha = 3.9$, and the centered forms ca. $\alpha = 3.4$. The precise inclusion of the latter methods is based by the fact that for small intervals the derivative may be monotonous, and thus its inclusion may be precise (up to the machine representation). Notice also that the linear least squares fit seems to be an informative description.

We summarize our results in detail in Table I and Table V. In Table I beside the values of α one can see the correlation coefficients in brackets that describe the strength of linear relationships (it is the square root of the variation ratio). In most cases the correlation coefficients indicate strong linear dependency. For most of the functions we have four lines, one for each examined set of boxes. These are the boxes extracted by the Moore-Skelboe B&B algorithm (denoted by B&B), the embedded sequence converging directly to a minimizer point (Opt), the random interval sequence (Rand), the upward going sequence (Up), and the union of the above series (All), respectively.

We have tested the standard global optimization test functions together with some others that are usually applied for investigating the efficiency of interval optimization methods [5]. These were Shekel 5, 7, 10, Hartman 3, 6, Goldstein-Price, Six-Hump-Camel-Back, Three-Hump-Camel-Back, Branin, Rosenbrock, Levy 3, 5, 8, 9, 10, 13, 14, 15, 16, Schwefel 21, 31, 25, 27, 218, 32, 37, and Ratz 4, 5. In some cases the very intensive computational requirements could not be met by standard setting. In such cases the results are missing for some interval sequences, as it is the case for the Griewank-5 test function.

The interesting results are **highlighted** in Table I. The empirical convergence speed of the natural extension is 2.766 for the Branin function on the set of boxes extracted by the B&B algorithm. It is better on this set than the central forms, and much better than the theoretical convergence order.

Table I.: The empirical convergence speed α with the correlation coefficient in brackets for the test functions with different sets of boxes.

Problem	Set	α				
name	name	Naive	Centered	Baumann	Slope.F	Slope.R
Branin	B&B	**2.766** (0.85)	1.545 (0.96)	1.521 (0.90)	2.782 (0.84)	2.811 (0.84)
	Opt	0.893 (0.41)	1.744 (0.89)	3.034 (0.95)	3.115 (0.88)	3.105 (0.88)
	Rand	0.943 (0.41)	2.298 (0.87)	1.144 (0.49)	2.953 (0.84)	2.964 (0.84)
	Up	0.796 (0.11)	3.850 (1.00)	3.872 (1.00)	3.793 (1.00)	3.856 (1.00)
	All	2.360 (0.73)	1.748 (0.90)	1.348 (0.53)	2.912 (0.86)	2.932 (0.86)
GP	B&B	1.209 (0.98)	3.365 (0.95)	3.388 (0.95)	3.931 (0.95)	3.952 (0.95)
	Opt	1.166 (0.98)	3.323 (0.96)	3.290 (0.96)	3.635 (0.89)	3.613 (0.89)
	Rand	1.064 (0.97)	2.427 (0.85)	1.974 (0.55)	3.576 (0.86)	3.610 (0.86)
	Up	**7.812** (0.97)	7.067 (0.99)	**7.833** (0.99)	**7.207** (0.97)	**7.590** (0.98)
	All	1.193 (0.98)	3.267 (0.94)	3.372 (0.93)	3.900 (0.94)	3.919 (0.94)
Griew5	B&B	1.661 (0.65)	1.916 (1.00)	1.790 (0.65)	1.876 (1.00)	1.876 (1.00)
L10	B&B	1.343 (0.54)	2.085 (1.00)	2.091 (1.00)	2.060 (1.00)	2.057 (1.00)
	Opt	1.032 (0.48)	2.070 (1.00)	2.082 (1.00)	2.038 (1.00)	2.047 (1.00)
	Rand	1.080 (0.88)	2.056 (0.86)	1.851 (0.64)	2.912 (0.85)	2.907 (0.85)
	All	1.076 (0.34)	2.062 (0.91)	2.015 (0.65)	2.553 (0.87)	2.550 (0.87)
L11	B&B	1.377 (0.54)	2.090 (1.00)	2.098 (1.00)	2.063 (1.00)	2.066 (1.00)
L12	B&B	1.383 (0.54)	2.090 (1.00)	2.102 (1.00)	2.071 (1.00)	2.068 (1.00)
L13	B&B	5.561 (0.82)	2.030 (1.00)	2.072 (1.00)	1.977 (1.00)	1.980 (1.00)
	Opt	4.893 (0.78)	1.994 (1.00)	2.051 (1.00)	1.954 (1.00)	1.955 (1.00)
	Rand	0.857 (0.96)	1.873 (0.99)	5.819 (0.83)	3.544 (0.75)	3.546 (0.75)
	All	1.724 (0.34)	1.885 (0.98)	5.058 (0.74)	3.138 (0.75)	3.140 (0.75)
L14	B&B	**2.191** (0.70)	2.028 (1.00)	2.043 (1.00)	2.002 (1.00)	2.008 (1.00)
	Opt	1.809 (0.65)	2.037 (1.00)	2.040 (1.00)	1.998 (1.00)	1.999 (1.00)
	Rand	0.944 (0.99)	2.139 (0.87)	2.331 (0.71)	3.128 (0.88)	3.130 (0.87)
	Up	**12.570** (0.63)	3.010 (0.99)	3.046 (0.99)	3.196 (0.98)	3.206 (0.98)
	All	0.826 (0.28)	2.097 (0.91)	2.468 (0.74)	2.737 (0.88)	2.740 (0.88)
L15	B&B	2.200 (0.70)	2.036 (1.00)	2.056 (1.00)	2.007 (1.00)	2.010 (1.00)
	Opt	1.826 (0.65)	2.037 (1.00)	2.052 (1.00)	2.008 (1.00)	2.009 (1.00)
	Rand	0.935 (0.99)	2.361 (0.89)	2.442 (0.72)	3.101 (0.86)	3.104 (0.86)
	All	1.295 (0.41)	2.227 (0.92)	2.368 (0.70)	2.686 (0.86)	2.689 (0.86)
L16	B&B	2.132 (0.69)	2.031 (1.00)	2.048 (1.00)	2.010 (1.00)	2.010 (1.00)
	Opt	1.706 (0.62)	2.027 (1.00)	2.042 (1.00)	2.000 (1.00)	2.000 (1.00)
	Rand	0.945 (0.99)	2.303 (0.89)	2.513 (0.73)	3.086 (0.87)	3.083 (0.87)
	Up	4.492 (0.19)	2.680 (0.97)	2.817 (0.98)	2.668 (0.95)	2.678 (0.96)
	All	0.983 (0.31)	2.179 (0.93)	2.482 (0.74)	2.607 (0.88)	2.605 (0.88)
L18	B&B	2.146 (0.69)	2.041 (1.00)	2.057 (1.00)	2.015 (1.00)	2.015 (1.00)
L3	B&B	0.967 (1.00)	3.070 (0.96)	3.073 (0.96)	3.495 (0.92)	3.494 (0.92)
	Opt	0.928 (0.99)	2.098 (0.68)	3.373 (0.91)	3.231 (0.87)	3.241 (0.87)
	Rand	1.068 (0.47)	2.127 (0.90)	2.286 (0.70)	2.902 (0.83)	2.918 (0.83)
	Up	**0.000** (0.76)	0.908 (0.99)	0.941 (0.98)	0.982 (0.98)	0.985 (0.99)
	All	1.042 (0.75)	2.761 (0.93)	3.133 (0.89)	3.281 (0.90)	3.284 (0.90)
L5	B&B	0.947 (0.99)	2.912 (0.97)	2.918 (0.97)	3.501 (0.94)	3.498 (0.94)
	Opt	0.939 (1.00)	2.591 (0.86)	3.118 (0.95)	3.240 (0.87)	3.249 (0.87)
	Rand	0.991 (1.00)	2.168 (0.89)	2.134 (0.67)	3.083 (0.86)	3.106 (0.87)
	Up	0.010 (0.21)	1.561 (0.99)	1.480 (0.99)	1.525 (0.99)	1.513 (0.99)
	All	0.948 (0.99)	2.286 (0.92)	3.007 (0.82)	3.085 (0.90)	3.081 (0.91)
L8	B&B	1.295 (0.53)	2.062 (1.00)	2.073 (1.00)	2.047 (1.00)	2.046 (1.00)
	Opt	1.019 (0.48)	2.050 (1.00)	2.060 (1.00)	2.034 (1.00)	2.039 (1.00)
	Rand	0.990 (0.99)	2.012 (0.87)	1.441 (0.56)	2.911 (0.87)	2.911 (0.87)
	Up	6.149 (0.24)	2.808 (0.98)	2.934 (0.99)	2.812 (0.98)	2.603 (0.97)
	All	0.715 (0.25)	2.037 (0.91)	1.959 (0.64)	2.620 (0.89)	2.619 (0.89)
L9	B&B	1.325 (0.54)	2.072 (1.00)	2.083 (1.00)	2.055 (1.00)	2.055 (1.00)
	Opt	1.018 (0.48)	2.064 (1.00)	2.075 (1.00)	2.037 (1.00)	2.039 (1.00)

Table I.: The empirical convergence speed α with the correlation coefficient in brackets for the test functions with different sets of boxes.

Problem name	Set name	α Naive	Centered	Baumann	Slope.F	Slope.R
	Rand	1.022 (0.47)	2.139 (0.86)	1.778 (0.63)	2.849 (0.84)	2.852 (0.85)
	All	1.040 (0.33)	2.109 (0.90)	1.951 (0.63)	2.550 (0.86)	2.551 (0.86)
Ratz4	B&B	0.983 (1.00)	2.880 (0.97)	2.981 (0.90)	0.523 (0.81)	1.004 (1.00)
	Opt	0.987 (1.00)	2.245 (0.94)	2.265 (0.94)	2.974 (0.86)	2.951 (0.85)
	Rand	1.240 (0.42)	2.131 (0.89)	1.763 (0.63)	2.833 (0.84)	2.824 (0.84)
	Up	**0.000** (0.76)	1.978 (1.00)	2.000 (1.00)	2.021 (1.00)	2.019 (1.00)
	All	1.313 (0.51)	2.166 (0.92)	2.292 (0.74)	2.847 (0.88)	2.839 (0.88)
RB	B&B	1.199 (0.46)	1.994 (1.00)	1.990 (1.00)	1.993 (1.00)	1.993 (1.00)
	Opt	0.767 (0.43)	2.005 (1.00)	1.995 (1.00)	1.992 (1.00)	2.003 (1.00)
	Rand	0.475 (0.14)	2.233 (0.88)	0.851 (0.34)	2.940 (0.80)	2.941 (0.80)
	Up	0.013 (0.33)	3.757 (0.99)	3.648 (0.99)	3.831 (1.00)	3.858 (0.99)
	All	0.472 (0.17)	2.199 (0.94)	1.707 (0.59)	2.473 (0.83)	2.474 (0.83)
S5	B&B	1.060 (1.00)	3.355 (0.92)	3.326 (0.92)	3.210 (0.89)	3.262 (0.90)
	Opt	0.997 (1.00)	2.275 (0.70)	3.281 (0.90)	3.220 (0.90)	3.223 (0.90)
	Rand	1.031 (0.99)	1.935 (0.90)	1.361 (0.56)	0.850 (0.41)	0.852 (0.41)
	Up	**0.000** (0.29)	1.984 (1.00)	2.019 (1.00)	0.062 (0.25)	0.058 (0.24)
	All	1.023 (0.98)	2.453 (0.89)	2.568 (0.77)	2.137 (0.70)	2.147 (0.70)
Sch218	B&B	2.044 (0.98)	2.744 (0.84)	2.784 (0.84)	2.846 (0.85)	2.846 (0.85)
	Opt	2.093 (1.00)	2.038 (1.00)	2.030 (1.00)	2.549 (0.89)	2.545 (0.89)
	Rand	0.746 (0.23)	2.104 (0.85)	0.866 (0.41)	2.774 (0.82)	2.772 (0.82)
	Up	1.994 (0.99)	2.004 (1.00)	1.954 (0.99)	2.045 (1.00)	2.017 (1.00)
	All	1.670 (0.66)	2.534 (0.83)	2.215 (0.72)	2.816 (0.85)	2.815 (0.85)
Sch21	B&B	2.043 (1.00)	2.072 (1.00)	2.068 (0.99)	2.068 (1.00)	1.066 (0.99)
	Opt	2.007 (1.00)	2.057 (1.00)	2.058 (1.00)	2.038 (1.00)	1.043 (0.99)
	Rand	1.027 (0.96)	2.112 (0.85)	1.204 (0.44)	3.171 (0.85)	1.026 (0.91)
	Up	2.179 (0.21)	8.154 (1.00)	8.074 (1.00)	8.140 (1.00)	8.087 (1.00)
	All	1.851 (0.90)	2.216 (0.94)	2.074 (0.76)	2.432 (0.89)	1.212 (0.88)
Sch25	B&B	1.933 (0.62)	2.000 (1.00)	2.009 (1.00)	2.012 (1.00)	2.011 (1.00)
	Opt	0.004 (0.24)	1.995 (1.00)	1.999 (1.00)	1.992 (1.00)	1.995 (1.00)
	Rand	0.429 (0.30)	1.952 (0.84)	0.447 (0.30)	2.818 (0.82)	2.823 (0.82)
	Up	1.442 (0.17)	1.992 (1.00)	1.960 (1.00)	2.002 (1.00)	1.972 (1.00)
	All	0.956 (0.37)	1.983 (0.94)	1.533 (0.52)	2.328 (0.85)	2.328 (0.85)
Sch31	B&B	2.024 (1.00)	2.041 (1.00)	2.025 (1.00)	2.034 (1.00)	2.032 (1.00)
	Opt	2.039 (1.00)	2.053 (1.00)	2.044 (1.00)	2.033 (1.00)	2.044 (1.00)
	Rand	0.768 (0.24)	1.872 (0.83)	1.235 (0.51)	2.959 (0.82)	2.962 (0.82)
	Up	**24.032** (0.63)	4.108 (0.97)	3.669 (0.98)	3.771 (0.95)	3.916 (0.97)
	All	1.057 (0.38)	1.952 (0.89)	1.712 (0.55)	2.657 (0.84)	2.661 (0.84)
Sch32	B&B	1.808 (0.61)	2.007 (1.00)	2.004 (1.00)	2.007 (1.00)	2.003 (1.00)
	Opt	0.824 (0.31)	2.012 (1.00)	2.013 (1.00)	1.998 (1.00)	1.991 (1.00)
	Rand	0.912 (0.28)	1.845 (0.85)	0.866 (0.34)	2.975 (0.85)	2.975 (0.85)
	Up	1.684 (0.34)	4.180 (1.00)	3.873 (1.00)	4.187 (1.00)	4.191 (1.00)
	All	0.868 (0.30)	2.067 (0.95)	1.867 (0.67)	2.479 (0.88)	2.476 (0.88)
$Sch37_{10}$	B&B	0.000 (0.00)	10.080 (1.00)	9.916 (1.00)	10.093 (1.00)	**10.103** (1.00)
$Sch37_{5}$	B&B	0.002 (0.13)	10.099 (1.00)	9.912 (1.00)	10.085 (1.00)	**10.108** (1.00)
	Opt	0.000 (0.02)	10.076 (1.00)	9.888 (1.00)	10.066 (1.00)	10.137 (1.00)
	Rand	0.000 (0.01)	1.863 (0.97)	0.795 (0.47)	2.201 (0.84)	2.204 (0.84)
	Up	0.001 (0.02)	10.089 (1.00)	10.039 (1.00)	10.039 (1.00)	10.104 (1.00)
	All	0.000 (0.03)	6.100 (0.74)	5.973 (0.75)	6.255 (0.76)	6.253 (0.76)
SHCB	B&B	1.033 (1.00)	2.985 (0.97)	3.257 (0.94)	3.421 (0.91)	3.407 (0.91)
	Opt	1.008 (1.00)	1.624 (0.88)	3.220 (0.94)	3.085 (0.86)	3.079 (0.86)
	Rand	1.005 (0.99)	2.215 (0.90)	1.696 (0.60)	3.021 (0.84)	3.008 (0.84)
	Up	4.029 (1.00)	6.046 (1.00)	5.918 (1.00)	6.031 (1.00)	5.968 (1.00)
	All	1.130 (0.96)	2.698 (0.93)	3.079 (0.83)	3.302 (0.89)	3.291 (0.89)

Table I.: The empirical convergence speed α with the correlation coefficient in brackets for the test functions with different sets of boxes.

Problem name	Set name	α Naive	Centered	Baumann	Slope.F	Slope.R
THCB	B&B	2.152 (1.00)	2.090 (1.00)	2.107 (1.00)	2.096 (1.00)	2.090 (1.00)
	Opt	2.072 (1.00)	2.035 (1.00)	2.028 (1.00)	2.040 (1.00)	2.049 (1.00)
	Rand	1.001 (0.99)	1.952 (0.84)	1.359 (0.48)	3.030 (0.83)	3.009 (0.82)
	Up	3.823 (0.99)	5.712 (1.00)	5.782 (1.00)	5.764 (0.99)	5.698 (0.99)
	All	1.528 (0.86)	2.215 (0.92)	2.349 (0.72)	2.883 (0.88)	2.878 (0.87)
All	All	2.026 (0.67)	2.818 (0.82)	2.843 (0.82)	3.237 (0.83)	3.121 (0.80)
	Opt	1.198 (0.37)	2.554 (0.64)	2.801 (0.68)	2.862 (0.68)	2.770 (0.66)
	Rand	0.975 (0.35)	2.125 (0.86)	1.607 (0.56)	2.887 (0.81)	2.684 (0.76)
	Up	2.686 (0.19)	4.672 (0.59)	4.651 (0.59)	4.479 (0.55)	4.465 (0.55)
	All	1.513 (0.51)	2.594 (0.82)	2.700 (0.72)	3.142 (0.83)	2.995 (0.79)

Examining the Goldstein-Price function, the upward going sequence gives interestingly bad results, although it fits well to earlier experiences that the minimization of this function is hard. The function Levy 14 (L14) is also interesting: the upward going sequences show a very bad inclusion, while the natural extension gives the best for the B&B sequence. The surprising results of the 5 and 10 dimensional functions Schwefel-37 ($\mathrm{Sch37}_5$ and $\mathrm{Sch37}_{10}$) are explained by the symbolic formulation of the function, $f(x) = \sum_{i=1}^{n} x_i^{10}$. Here the zero convergence speed of the natural extension should be understood as ∞, because the related inclusion is always exact, what gives a horizontal line at -30.

At the end of the table one can read the empirical convergence speed generalized to all functions. The last line shows the empirical convergence orders for all the functions on all the boxes. These values are larger than the respective theoretical convergence orders. This is to be explained by the fact, that the empirical convergence speed measures the average speed, while the theoretical convergence order gives a lower bound.

The results for parameter c are less informative, but still the values are important to find out which inclusion function is the most advantageous for a given size of interval. The Table V can be found in the appendix.

We plane to examine also some more sophisticated inclusion functions with the described technique to be able to compare them. These are for example the affine arithmetic [11, 17], and the Taylor forms [3, 13].

6.1. Evaluation and discussion

For the usage of inclusion functions in numerical algorithms the most important question is which inclusion function provides the most precise lower and upper bounds. We have considered this question within

Table II. Test functions without intersecting best inclusion functions in the interval $[10^{-12}, 10]$.

Naive	B&B:	L8, L9, L10, L11, L12, L13, L14, L15, L16, L18, RB, Sch21, Sch31, Sch32, Sch25
	Opt:	L10, L13, L14, L15, L16, L8, L9, RB, Sch21, Sch31, Sch32, Sch25
	Up:	GP, L16, Sch31, Sch25
	All:	Sch25
Baumann	Rand:	L8, L9, L10, L5, L3, L16, L15, L14, L13, SCHB
	All:	Sch31, Ratz4, SHCB, L5
Rev.Slope	Opt:	L5, SHCB
	B&B:	Sch218
Forw.Slope	Up:	Sch21

a scenario similar to that we can meet when solving standard global optimization test problems.

The rule of thumb folklore in interval mathematics state that for intervals with a width larger than one one should use natural interval extension, while for smaller intervals a centered form or a slope based inclusion function.

In our investigation we have determined that inclusion functions for a certain problem and interval sequence which have the minimal overestimation for an interval size of 10^{-12} and 10, respectively. This was made on the basis of the data collected by linear regression (Tables I and V). The cases when one inclusion function is minimal everywhere are shown in Table II. In case these inclusion functions were different, we have also calculated the intersection point of the related lines, i.e. the transition point at which the user should switch from the second to the first inclusion function to have the better precision. According to our experiences, these bounds are realistic when solving usual global optimization problems.

The obtained results are comprised in Table III. Check first that line which contains the solution of the Goldstein-Price problem (Rev.Slope–Naive, B&B), also illustrated on Figure 1. As it can be read on the bottom right graph, the best inclusion function is for small intervals the reverse slope arithmetic, while for larger intervals the natural interval extension provides the most precise inclusion (the later is denoted in Table III as Naive). These optimal inclusion functions for the GP, B&B case, and also the transition point of 0.2114 are realistic, they are according to our expectations.

Table III. The two most precise inclusion functions in the interval $[10^{-12}, 10]$ together with the transition point.

Baumann–Naive	B&B:	Griew5 ($3.0243 \cdot 10^{-8}$)
	All:	L10 (0.245), L13 (1.074), L14 (0.0427), L15 (1.232), L16 (0.0005118), L8 (6.893), L9 (0.540), S5 (0.157)
	Up:	S5 (0.2985)
Centered–Naive	B&B:	S5 (0.0867)
	Up:	L5 (2.1007)
Forw.Slope–Naive	B&B:	L3 (2.1887), L5 (1.7504), SHCB (3.4291)
	Opt:	GP (3.0196), Branin ($8.794 \cdot 10^{-10}$)
	Up:	Sch218 (0.424), Ratz4 (1.105), SHCB (5.811), THCB (6.229)
	All:	THCB ($7.4389 \cdot 10^{-10}$)
Rev.Slope–Naive	B&B:	Sch37_{10} (0.001338), Sch37_5 (0.001397), GP (0.2114)
	Rand:	Sch25 ($1.237 \cdot 10^{-9}$)
	Up:	L3 (0.0998), Sch32 ($1.447 \cdot 10^{-9}$), RB ($7.931 \cdot 10^{-9}$)
	All:	L3 (1.477), RB ($1.03511 \cdot 10^{-10}$)
Forw.Slope–Baumann	Rand:	Ratz4 ($1.057 \cdot 10^{-12}$), Sch21 ($1.394 \cdot 10^{-9}$), Sch218 ($7.150 \cdot 10^{-10}$), GP ($9.6332 \cdot 10^{-10}$)
	All:	Sch21 ($1.5586 \cdot 10^{-9}$), Sch218 ($4.8106 \cdot 10^{-8}$)
Rev.Slope–Baumann	Rand:	Branin ($1.597 \cdot 10^{-11}$), Sch31 ($2.3726 \cdot 10^{-12}$), THCB ($3.284 \cdot 10^{-9}$), Sch32 ($4.198 \cdot 10^{-9}$), RB ($7.468 \cdot 10^{-10}$)
	All:	Branin (0.01046), THCB ($9.087 \cdot 10^{-10}$)
Forw.Slope–Rev.Slope	Opt	Sch218 (0.001778), Ratz4 ($3.675 \cdot 10^{-5}$)
Naive–Rev.Slope	Up	L14 (0.1247)
	B&B	THCB ($3.535 \cdot 10^{-7}$)
Naive–Forw.Slope	Up	L8 (5.124)
	Opt	THCB (8.058)
Baumann–Forw.Slope	B&B	Ratz4 (0.3479)
	All	S5 ($3.990 \cdot 10^{-8}$)
	Rand	S5 ($5.823 \cdot 10^{-8}$)
Rev.Slope–Centered	Opt	L3 (1.772)
Rev.Slope–Forw.Slope	B&B	Branin ($1.1721 \cdot 10^{-10}$)

Table IV. The two most precise inclusion functions generalized to all of the function in the interval $[10^{-12}, 10]$.

Forw.Slope– Naive	B&B (0.05309), Opt ($1.919{\cdot}10^{-7}$), All (0.006012)
Forw.Slope– Baumann	Rand ($1.21444{\cdot}10^{-12}$)
Centered– Naive	Up ($1.22637{\cdot}10^{-9}$)

Out of the over 110 cases solved successfully, in closely 50 instances a single inclusion function proved to be the best for the whole range studied. It was mostly the natural interval extension, which is to be explained by the relative simple problems. For small intervals the monotonicity makes the natural interval extension precise (before outward rounding).

There is a big variety of pairs of optimal inclusions, still for the most important branch-and-bound sequence the naive arithmetic (as single choice) is most often the best. For the increasing interval sequence (denoted by Up) the present bounds of 10^{-12} and 10 are not always realistic, it is also reflected by the fact that in many cases the naive interval arithmetic was the best for the larger bound. It is in a good accordance with the experiences again.

Beyond the detailed results for the test problems, Table IV gives the most informative results comprising the knowledge collected on singular interval sequences. According to the average behaviour, when inclusion functions are used in a branch-and-bound method for optimization, then naive arithmetic is the best choice for intervals with width larger than 0.05, while the forward mode slope is the best for smaller intervals. The same pair of inclusion function is to be selected for an embedded sequence of intervals converging to a global minimizer point, and it holds also for the total set of intervals investigated (with different transition points). For random intervals the forward slope and the Baumann centered form is optimal (regarding the intersection point, better only the second one). In a similar way, for larger and increasing size interval sequences the naive arithmetic is the best.

An anonymous referee asked us to provide a detailed comparison with the similar results in [15]. Here the only comparable data are on a one-dimensional function, $(x^2 - 2x + 2)e^x$. The authors provide relative overestimation values for the intervals $[1 - r, 1 + r]$, where r is between 0.002 and 30. The conclusion is that the second order Taylor form gives the best inclusion for small intervals ($r \leq 0.8$), while for larger intervals the natural interval extension is the best. The overestimation figures indicate (as it can be expected) closely linear convergence for the natural interval extension, and seemingly quadratic for the Taylor form.

We have repeated our empirical convergence speed determination protocol on this problem. Although 1, the center of the intervals studied in [15] is not an optimizer point (the function does not have such, only a stationary point at zero), we have investigated interval sequences converging to 1. We have also found that the natural interval extension has an empirical convergence order close to one (1.07), and the other studied inclusion functions were closely quadratic (with empirical convergence order between 1.728 and 3.154). According to our results, the Baumann centered form proved to be the best for r values in $[10^{-12}, 4]$. Let us mention that this inclusion function was in all but a few cases closely exact due to the monotonicity properties of the function. With randomly chosen interval sequences we have arrived to basically the same conclusion. In our case the sophisticated inclusion functions based on first order derivative information have outperformed the natural interval extension.

7. Summary

The introduced empirical convergence speed values for the examined inclusion functions on the test functions allow a deeper understanding of the behaviour of overestimation. On this basis we can answer the question: how to provide a usable formula to choose the best inclusion function for a given function at a given state of the optimization. According to the present study for optimization the best is to use natural interval extension when the actual interval width is larger than 0.5, and forward mode slope arithmetic for smaller intervals. The presented methodology is obviously applicable also for other verified numerical algorithms.

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Appendix

Table V.: The logarithm of the empirical convergence parameter c for the test functions for some interval sequences.

Problem name	Set name	$\log_{10} c$				
		Naive	Centered	Baumann	Slope.F	Slope.R
BR	B&B	-0.904	-0.898	-1.478	-4.635	-4.347
	Opt	-22.132	0.009	2.117	-2.010	-2.112
	Rand	-2.521	0.704	-20.899	-1.341	-1.249
	Up	6.035	-1.560	-1.958	-1.969	-2.183
	All	-2.075	-0.050	-6.166	-3.227	-3.029
GP	B&B	4.708	6.684	6.513	6.404	6.559
	Opt	4.363	7.626	7.419	3.178	3.015
	Rand	5.185	6.464	-8.748	5.696	6.125
	Up	2.680	4.644	3.683	3.509	2.998
Griew5	B&B	-17.372	-0.855	-16.402	-1.042	-1.045
L10	B&B	-19.563	0.032	-0.176	-0.507	-0.536
	Opt	-21.820	-0.103	-0.284	-0.683	-0.637
	Rand	0.497	1.240	-15.008	0.597	0.586
	All	-8.827	0.678	-8.254	0.103	0.092
L11	B&B	-19.349	0.152	-0.040	-0.398	-0.376
L12	B&B	-19.286	0.221	0.045	-0.293	-0.303
L13	B&B	-8.347	1.539	1.418	1.001	1.004
	Opt	-10.498	1.713	1.576	1.144	1.199
	Rand	1.055	2.784	-4.352	3.594	3.543
	All	-2.277	2.397	-2.380	2.822	2.797
L14	B&B	-12.439	1.546	1.407	0.992	1.016
	Opt	-15.293	1.396	1.166	0.761	0.789
	Rand	1.320	3.394	-10.031	3.146	3.159
	Up	7.545	0.837	0.732	-0.902	-0.921
	All	-6.901	2.720	-4.653	2.128	2.147
L15	B&B	-12.395	1.567	1.452	0.999	1.021
	Opt	-15.155	1.427	1.292	0.815	0.859
	Rand	1.242	3.730	-9.712	2.638	2.655
	All	-4.829	2.804	-4.926	1.956	1.979
L16	B&B	-12.926	1.527	1.388	0.996	0.993
	Opt	-16.102	1.403	1.235	0.807	0.829
	Rand	1.197	3.445	-8.837	2.901	2.900
	Up	-17.245	1.690	1.400	0.382	0.359
	All	-8.213	2.540	-3.280	1.855	1.855
L18	B&B	-12.827	1.581	1.453	1.022	1.029
L3	B&B	2.078	3.305	3.202	1.218	1.238
	Opt	1.970	-0.261	2.150	-0.633	-0.545
	Rand	-0.406	2.802	-10.990	2.313	2.416

Table V.: The logarithm of the empirical convergence parameter c for the test functions for some interval sequences.

Problem name	Set name	$\log_{10} c$				
		Naive	Centered	Baumann	Slope.F	Slope.R
	Up	1.783	3.251	3.195	2.814	2.769
	All	1.707	3.141	1.244	1.295	1.327
L5	B&B	1.951	2.602	2.546	1.330	1.337
	Opt	2.028	3.539	4.723	-0.539	-0.585
	Rand	1.722	2.845	-12.358	1.973	2.402
	Up	2.379	1.879	1.971	1.681	1.723
	All	1.614	2.531	-2.892	1.063	1.208
L8	B&B	-19.890	-0.161	-0.320	-0.647	-0.646
	Opt	-21.929	-0.461	-0.650	-0.922	-0.912
	Rand	0.361	1.039	-18.057	1.192	1.197
	Up	-3.459	-0.049	-0.333	-1.091	-0.614
	All	-8.916	0.703	-9.959	0.415	0.415
L9	B&B	-19.687	-0.073	-0.253	-0.579	-0.582
	Opt	-21.940	-0.259	-0.439	-0.804	-0.805
	Rand	-2.261	0.943	-15.611	0.330	0.337
	All	-9.588	0.515	-9.344	-0.036	-0.034
Ratz4	B&B	-0.338	2.318	0.818	-0.309	1.492
	Opt	-0.410	0.943	0.468	-3.752	-3.854
	Rand	-7.741	-0.797	-15.403	-2.589	-2.654
	Up	0.140	0.602	0.522	0.052	0.041
	All	-6.165	-0.436	-10.184	-2.719	-2.771
RB	B&B	-14.180	2.496	2.082	2.140	2.147
	Opt	-23.624	2.671	2.567	2.180	2.259
	Rand	-15.186	2.864	-18.302	0.774	0.773
	Up	-29.934	2.055	2.022	1.290	1.213
	All	-18.107	3.409	-5.726	1.875	1.883
S5	B&B	-0.953	1.484	1.182	-0.026	0.411
	Opt	-1.047	-1.141	0.789	0.420	0.420
	Rand	-2.279	-1.689	-18.933	-22.630	-22.614
	Up	1.172	2.677	2.232	2.407	2.420
	All	-1.863	-0.176	-7.875	-11.064	-10.963
Sch218	B&B	-0.025	-2.615	-3.145	-3.834	-3.849
	Opt	-0.517	-0.272	-0.318	-1.623	-1.634
	Rand	-12.748	-0.668	-20.612	-3.162	-3.168
	Up	-0.954	-0.446	-0.439	-0.935	-0.835
	All	-3.681	-2.025	-7.967	-3.569	-3.581
Sch21	B&B	0.869	1.850	1.482	1.435	2.006
	Opt	0.454	1.857	1.685	1.313	1.879
	Rand	1.578	3.275	-15.431	1.988	3.644
	Up	3.184	-2.018	-2.027	-3.094	-2.750
	All	1.150	3.103	-0.655	2.498	3.432

Table V.: The logarithm of the empirical convergence parameter c for the test functions for some interval sequences.

Problem name	Set name	$\log_{10} c$				
		Naive	Centered	Baumann	Slope.F	Slope.R
Sch25	B&B	-8.036	0.877	0.548	0.531	0.532
	Opt	-29.914	0.974	0.799	0.602	0.601
	Rand	-26.431	0.460	-26.354	-0.339	-0.318
	Up	-21.594	0.927	0.822	0.489	0.577
	All	-18.537	0.736	-9.189	0.116	0.117
Sch31	B&B	0.088	1.232	0.990	0.795	0.868
	Opt	-0.004	1.174	0.998	0.711	0.802
	Rand	-8.985	1.852	-19.672	0.385	0.404
	Up	-72.693	-0.170	0.128	-0.165	-0.313
	All	-6.891	1.756	-10.592	0.759	0.809
Sch32	B&B	-7.611	0.948	0.570	0.588	0.559
	Opt	-17.520	0.834	0.600	0.394	0.359
	Rand	-11.403	0.421	-17.429	0.241	0.238
	Up	-23.546	-0.758	-0.410	-1.380	-1.385
	All	-13.510	1.617	-3.991	1.196	1.177
$\mathrm{Sch37}_{10}$	B&B	-29.902	0.081	-1.781	-0.979	-0.870
$\mathrm{Sch37}_{5}$	B&B	-29.877	-0.054	-1.925	-1.171	-1.026
	Opt	-29.887	-0.221	-2.239	-1.260	-0.756
	Rand	-29.900	-5.581	-20.629	-5.973	-5.944
	Up	-29.896	-0.941	-1.374	-1.818	-1.988
	All	-29.895	2.532	-2.707	1.548	1.558
SHCB	B&B	1.261	2.735	1.699	-0.017	-0.175
	Opt	1.078	-0.242	2.793	-2.258	-2.365
	Rand	1.312	1.868	-16.069	1.181	1.050
	Up	-0.333	-1.022	-1.274	-1.863	-1.683
	All	1.986	2.969	-1.954	0.779	0.646
THCB	B&B	1.663	1.684	1.429	1.317	1.263
	Opt	0.939	1.361	1.117	0.968	1.040
	Rand	2.075	1.401	-14.115	0.204	-0.117
	Up	0.624	0.081	-0.504	-0.918	-0.783
	All	2.613	2.941	-2.968	1.954	1.815
All	B&B	1.418	3.790	3.526	2.962	3.038
	Opt	-11.417	0.877	1.115	-0.240	-0.186
	Rand	-3.986	1.655	-15.472	-0.220	-0.091
	Up	-17.725	-0.027	-0.255	-0.808	-0.766
	All	-1.558	3.267	-1.153	2.060	2.165