Applications of Linear Programming

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Lecture 1

Why LP?

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- Industries that use linear programming models include transportation, energy, telecommunications, and manufacturing
- Useful in modeling diverse types of problems in, for instance
  - planning
  - routing
  - scheduling
  - assignment

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- 1979 – *Khachiyan*'s polynomial time algorithm
- 1984 – *Karmarkar*'s breakthrough: interior-point method for solving LP

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- GE credit card payment system: Makuch et al (1989)

Some real applications

- 2013: Dutch Delta Program Commissioner used mixed integer nonlinear programming to derive an optimal investment strategy for strengthening dikes for protection against high water and keeping freshwater supplies up to standard, resulting in savings of 8 billion euros in investment costs

Some real applications

- 2011: Midwest Independent Transmission System Operator used mixed integer programming to determine when each power plant should be on or off, the power plant output levels and prices to minimize the cost of generation, start-up and contingency reserves for 1000 power plants with total capacity of 146,000 MW spread over 13 Midwestern states of U.S. and Manitoba (Canada) owned by 750 companies supplying 40 million users, resulting in savings of $2 billion over the period 2007-2010.

Some real applications

- 2008: Netherlands Railways developed a constraint programming based railway timetable for scheduling about 5,500 trains daily, while ensuring maximum utilization of railway network, improving the robustness of the timetable, and optimal utilization of rolling stock and crew thereby resulting in additional annual profit of 40 euros million.

LP standard form

Linear program (LP) in a **standard form** (maximization)

$$
\begin{array}{lcccccccccl}
\max & c_1x_1 & + & c_2x_2 & + & \dots & + & c_nx_n & & & \text{Objective function} \\
\text{subject to} & a_{11}x_1 & + & a_{12}x_2 & + & \dots & + & a_{1n}x_n & \leq & b_1 & \\
 & a_{21}x_1 & + & a_{22}x_2 & + & \dots & + & a_{2n}x_n & \leq & b_2 & \\
 & \vdots & + & \vdots & & & & \vdots & & \vdots & \text{Constraints} \\
 & a_{m1}x_1 & + & a_{m2}x_2 & + & \dots & + & a_{mn}x_n & \leq & b_m & \\
 & & & & & & & x_1, x_2, \dots, x_n & \geq & 0 & \text{Sign restrictions}
\end{array}
$$

Feasible solution (point) $P = (p_1, p_2, \dots, p_n)$ is an assignment of values to the $p_1, \dots, p_n$ to variables $x_1, \dots, x_n$ that satisfies **all** constraints and **all** sign restrictions.

Feasible region $\equiv$ the set of all feasible points.

Optimal solution $\equiv$ a feasible solution with maximum value of the objective function.

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Product mix

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- To make one toy soldier costs \$10 for raw materials and \$14 for labor; it takes 1 hour in the carpentry shop, and 2 hours for finishing. To make one train costs \$9 for raw materials and \$10 for labor; it takes 1 hour in the carpentry shop, and 1 hour for finishing.

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- There are 80 hours available each week in the carpentry shop, and 100 hours for finishing. Each toy soldier is sold for \$27 while each train for \$21. Due to decreased demand for toy soldiers, the company plans to make and sell at most 40 toy soldiers; the number of trains is not restriced in any way.

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What is the optimum (best) product mix (i.e., what quantities of which products to make) that maximizes the profit (assuming all toys produced will be sold)?

Product mix: LP formulation

Decision variables:

- x_1= # of toy soldiers
- x_2= # of toy trains

Objective: maximize profit

- $\$27 - \$10 - \$14 = \$3$ profit for selling one toy soldier $\Rightarrow 3x_1$ profit (in \$) for selling x_1 toy soldier
- $\$21 - \$9 - \$10 = \$2$ profit for selling one toy train $\Rightarrow 2x_2$ profit (in \$) for selling x_2 toy train

$\Rightarrow \underbrace{z = 3x_1 + 2x_2}_{\textit{objective function}}$ profit for selling x_1 toy soldiers and x_2 toy trains

Constraints:

- producing x_1 toy soldiers and x_2 toy trains requires
  (a) $1x_1 + 1x_2$ hours in the carpentry shop; there are 80 hours available
  (b) $2x_1 + 1x_2$ hours in the finishing shop; there are 100 hours available
- the number x_1 of toy soldiers produced should be at most 40

Variable domains: the numbers x_1, x_2 of toy soldiers and trains must be non-negative (**sign restriction**)

$$\begin{array}{rrcrcl}
\text{Max} & 3x_1 & + & 2x_2 & & \\
& x_1 & + & x_2 & \leq & 80 \\
& 2x_1 & + & x_2 & \leq & 100 \\
& x_1 & & & \leq & 40 \\
& & & x_1, x_2 & \geq & 0
\end{array}$$

A more complicated example

You have \$100. You can make the following three types of investments:

Investment A. Every dollar invested now yields \$0.10 a year from now, and \$1.30 three years from now.

Investment B. Every dollar invested now yields \$0.20 a year from now and \$1.10 two years from now.

Investment C. Every dollar invested a year from now yields \$1.50 three years from now.

During each year leftover cash can be placed into money markets which yield 6% a year. The most that can be invested a single investment (A, B, or C) is \$50.

Formulate an LP to maximize the available cash three years from now.

Example: LP formulation

Decision variables: x_A, x_B, x_C, amounts invested into Investments A, B, C, respectively
y_0, y_1, y_2, y_3 cash available/invested into money markets now, and in 1,2,3 years.

$$
\begin{array}{lccccccccc}
\text{Max} & y_3 & & & & & & & & \\
\text{s.t.} & x_A & + & x_B & & & + & y_0 & = & 100 \\
 & 0.1x_A & + & 0.2x_B & - & x_C & + & 1.06y_0 & = & y_1 \\
 & & & 1.1x_B & & & + & 1.06y_1 & = & y_2 \\
 & 1.3x_A & & & + & 1.5x_C & + & 1.06y_2 & = & y_3 \\
 & x_A & & & & & & & \leq & 50 \\
 & & & x_B & & & & & \leq & 50 \\
 & & & & & x_C & & & \leq & 50 \\
 & & & & & & & x_A, x_B, x_C, y_0, y_1, y_2, y_3 & \geq & 0
\end{array}
$$

Example: let us see what is going on

Activities

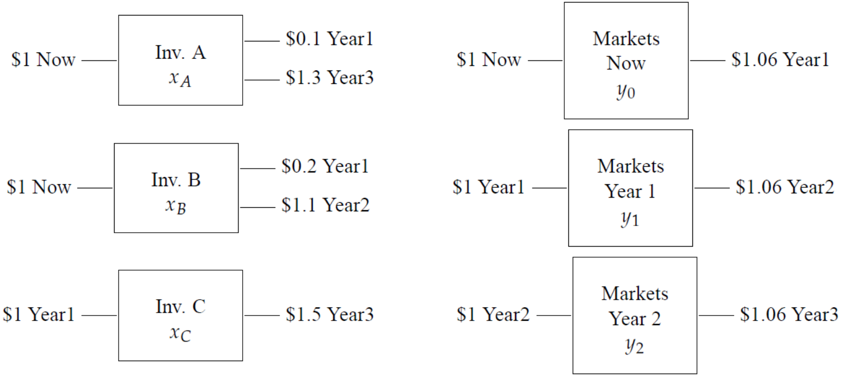
Items		Inv. A	Inv. B	Inv. C	Markets Now	Markets Year 1	Markets Year 2		External flow
	Now	-1	-1		-1			=	-100
	Year1	0.1	0.2	-1	1.06	-1		=	0
	Year2		1.1			1.06	-1	=	0
	Year3	1.3		1.5			1.06		maximize

Sign convention: inputs have **negative** sign, outputs have **positive** signs.

External in-flow has **negative** sign, external out-flow has **positive** sign.

We have in-flow of \$100 cash "Now" which means we have $-\$100$ on the right-hand side. No in-flow or out-flow of any other item.

Example: let us see what is going on

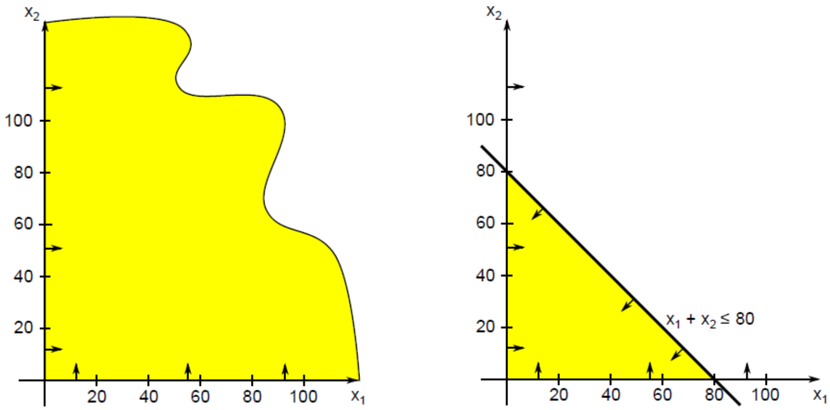


$$
\begin{array}{llll}
\text{Max} & 1.3x_A + 1.5x_C + 1.06y_2 & & \\
\text{s.t.} & x_A + x_B + y_0 & = & 100 \\
 & 0.1x_A + 0.2x_B - x_C + 1.06y_0 - y_1 & = & 0 \\
 & 1.1x_B + 1.06y_1 - y_2 & = & 0 \\
 & y_0,\ y_1,\ y_2 & \geq & 0 \\
 & 0 \leq x_A,\ x_B,\ x_C & \leq & 50
\end{array}
$$

Product mix: graphical method

1. Find the feasible region.

- Plot each constraint as an equation ≡ line in the plane
- Feasible points on one side of the line – plug in (0,0) to find out which



Start with $x_1 \geq 0$ and $x_2 \geq 0$

add $x_1 + x_2 \leq 80$

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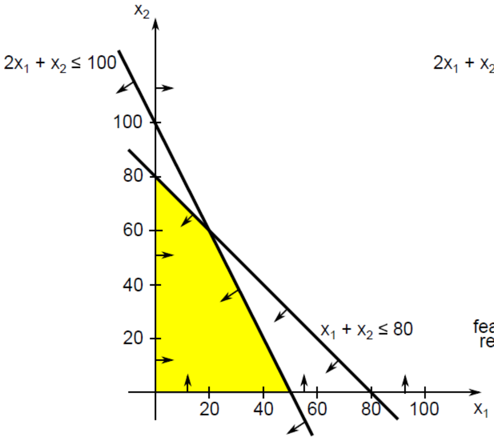
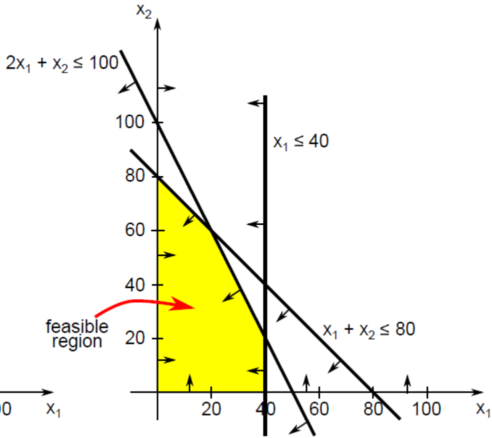


add $2x_1 + x_2 \leq 100$



add $x_1 \leq 40$

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Theorem 2. *Every linear program has either*

1. *a* **unique** *optimal solution, or*
2. *multiple (***infinity***) optimal solutions, or*
3. *is* **infeasible** *(has no feasible solution), or*
4. *is* **unbounded** *(no feasible solution is maximal).*

In higher dimensions...

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R^n: n-dimensional **linear space** over the real numbers – elements: real **vectors** of n elements

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E^n: n-dimensional **Euclidean space**, with an **inner product** operation and a **distance** function are defined as follows

- $\langle x, y \rangle = x^T y = x_1 y_1 + x_2 y_2 + \ldots + x_n y_n$, $||x|| = \sqrt{\langle x, x \rangle}$ norm
- $d(x, y) = ||x - y||_2 = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \ldots + (x_n - y_n)^2}$

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This distance function is called the Euclidean metric. This formula expresses a special case of the *Pythagorean theorem*.

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n-dimensional hyperplane:

$$\{ \, x \; : \; x \in E^n, \; a_1 x_1 + a_2 x_2 + \ldots + a_n x_n = b \, \},$$

where $a_1, a_2, \ldots, a_n, b \in \mathbb{R}$ given (fixed) numbers

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Feasible region $\leftrightarrow$ Intersection of half-spaces (and hyperplanes)

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Theorem 1. tells us that a linear objective function achieves its maximal value (if exists) is a corner (extreme) point of the feasible region (i.e. polytope).

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$\Longrightarrow$ **Simplex algorithm** (see Lecture 2)