Környezetfüggetlen rendtípusok

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Roadmap

- Order types, well-orderings, ordinals and scattered order types
- Some applications of these constructs in computer science
- Order types of regular and context-free languages
- Known results regarding decidability and complexity issues
- Open questions of the area
- Some proof techniques

Ordered sets

A (linearly/totally) ordered set is a pair $(X, <)$ with X being a set and $<$ being a total order on X: an irreflexive, transitive and trichotomous relation

In computer science we are usually only interested in countable sets

since we want to represent their elements by a finite amount of information

Examples

- The set $\mathbb{N} = \{0, 1, 2, \ldots\}$ of natural numbers, equipped with their standard ordering
- The set $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$ of integers ($+$ their standard order)
- The set $\{-4, -2, 0, 2, 4, \ldots\}$ of the even integers
- The set $\mathbb{Q}$ of rationals
- The set $\{0, 1\}^*$ of finite binary strings, ordered lexicographically

Order types

Amongst these, there is an order-preserving bijection between the integers and the even integers

This is an equivalence relation over all the possible orderings

The classes of this equivalence relation are called order types.

The order type. . .

- of the natural numbers is denoted by ω
- of the integers (and of the even integers) is denoted by ζ
- of the rationals is denoted by η
- of the finite sets is denoted by their cardinality

e.g. the order type sun $<$ mon $<$ tue $< \ldots <$ sat of the days of the week is denoted by 7

these order types are pairwise different, e.g. $\omega \neq \zeta$

Sums of orderings, order types

If $(X,<)$ and $(Y,\prec)$ are ordered sets of order type o_X and o_Y, respectively, then the order type of their (disjoint) union $X \times \{0\} \cup Y \times \{1\}$, ordered by

- each element of X is smaller than each element of Y,
- inside X and Y, the elements are ordered according to the original $<$ and $\prec$, resp,

is usually denoted by $o_X + o_Y$.

Two copies of the natural numbers, placed next to each other:

$$(0,0) < (1,0) < (2,0) < \ldots < (0,1) < (1,1) < (2,1) < \ldots$$

has order type $\omega + \omega$.

$\omega + \omega \neq \omega$	$\eta + \eta = \eta$
$\omega + 1 \neq \omega$	$1 + \omega = \omega$

"Infinite" sums of orderings, order types

If $I = (I, <)$ is an "indexing" ordering and for each $i \in I$, $X_i = (X_i, <_i)$ is an ordering, then $\sum_{i \in I} X_i$ is the ordering with

- domain $\bigcup_{i \in I} X_i \times \{i\}$
- equipped with the anti-lexicographic ordering: $(p, i) < (q, j)$ if and only if either $i < j$ or ($i = j$ and $p <_i q$)

The order type is denoted $\sum_{i \in I} o(X_i)$.

We can place a number of orderings, each being either of type 1 or of type ω, next to each other, indexed by ω and we can get e.g.:

- $1 + 1 + 1 + 1 + 1 + \ldots = \omega$
- $1 + 1 + 1 + 1 + \omega + 1 + 1 + 1 + 1 + \ldots = \omega + \omega$
- $1 + \omega + 1 + 1 + \omega + 1 + 1 + 1 + \omega + \ldots = \omega + \omega + \omega + \ldots$

Products of orderings, order type

If we have a sum of the form $\sum_{i \in I} X_i$ with each X_i having the same order type o, then the order type of this sum is also denoted by $o \times o(I)$.

$$\omega + \omega + \omega + \ldots = \omega \times \omega$$
$$\omega + 1 + \omega + 1 + \omega + 1 + \ldots = (\omega + 1) \times \omega = \omega \times \omega$$
$$\omega + \omega = \omega \times 2$$
$$2 \times \omega = 2 + 2 + 2 + \ldots = \omega$$
$$\ldots + \omega + \omega + \omega + \ldots = \omega \times \zeta$$
$$\zeta + \zeta + \zeta + \ldots = \zeta \times \omega \neq \omega \times \zeta$$

Well-orderings, ordinals

- An ordering is a well-ordering if it contains no infinite descending chain
- The order types of well-orderings are called ordinals

- the finite order types 0, 1, 2, ... are ordinals, as well as ω
- ζ and η are not ordinals
- $\omega \times \omega$ is an ordinal

The order of the ordinals

- the (countable) ordinals themselves are also ordered: $o_1 \preceq o_2$ if some ordering of type o_1 can be mapped into an ordering of type o_2 in an order-preserving way
- if not the other way around: $o_1 \prec o_2$
  - $\omega \prec \omega + 1$
  - $\omega + \omega \prec \omega \times \omega$
- turns out $\prec$ is a total ordering over the ordinals: for each pair o_1, o_2 of ordinals, exactly one of $o_1 \prec o_2$, $o_2 \prec o_1$ or $o_1 = o_2$ holds
- (this is not true for all the order types, e.g. $\eta + 1 + 1 + \eta$ can be embedded into η and vice versa but they are not the same)
- moreover, this $\prec$ contains no infinite descending chains $\Rightarrow$ the ordinals themselves are also well-ordered

Successor and limit ordinals

An ordinal α is either...

- a successor ordinal, that is, $\alpha = \beta + 1$ for some (smaller) ordinal β,
- or a limit ordinal, that is, $\alpha = \bigvee_{\beta \prec \alpha} \beta$ is the supremum of all the ordinals smaller than α

- $7 = 6 + 1$ is a successor ordinal
- ω is a limit ordinal: it is the supremum of $\{0, 1, 2, 3, \ldots\}$
- $\omega + 3 = (\omega + 2) + 1$ is a successor ordinal
- $\omega + \omega$ and $\omega \times \omega$ are limit ordinals
- $0 = \bigvee \emptyset$ is a limit ordinal (usually treated separately in proofs)

Exponentation of ordinals

If α and β are ordinals, then α^β is...

- 1 if $\beta = 0$,
- $(\alpha^\gamma) \times \alpha$ if $\beta = \gamma + 1$ is a successor ordinal,
- 0 if β is a limit ordinal and $\alpha = 0$,
- $\bigvee_{\gamma \prec \beta} \alpha^\gamma$ if β is a limit ordinal and $\alpha \neq 0$.

- $\omega^1 = (\omega^0) \times \omega = 1 \times \omega = \omega$
- $\omega^2 = (\omega^1) \times \omega = \omega \times \omega$
- $\omega^3 = \omega \times \omega \times \omega$ associative
- $\omega^\omega = \bigvee_{n < \omega} \omega^n = 1 + \omega + \omega^2 + \omega^3 + \dots$

Cantor normal form

Each ordinal α can be uniquely written as a finite sum of the form

$$\alpha = \omega^{\alpha_1} \times n_1 + \omega^{\alpha_2} \times n_2 + \ldots + \omega^{\alpha_k} \times n_k$$

for some integer $k \geq 0$, ordinals $\alpha_1 > \alpha_2 > \ldots > \alpha_k$ and integer coefficients $n_1, \ldots, n_k > 0$.

- $\omega^2 + \omega^2 + \omega + \omega + \omega + 2 = \omega^2 \times 2 + \omega \times 3 + 2$
- $\omega + \omega^2 + \omega + \omega^2 + \omega = \omega^2 \times 2 + \omega$
- $\omega^\omega \times (\omega + 1) = \omega^{\omega+1} + \omega^\omega$

seems like a finitely presentable normal form for ordinals

$$\epsilon_0 = 1 + \omega + \omega^\omega + \omega^{\omega^\omega} + \omega^{\omega^{\omega^\omega}} + \ldots$$

Then, $\epsilon_0 = \omega^{\epsilon_0}$.

this guy is still countable

Applications of ordinals

Halting conditions

Since the ordinals themselves are well-ordered, if we can assign an ordinal to each program state such that the ordinal decreases in each step, we proved termination.

Ackermann function

$$A(n,m) := \begin{cases} m+1 & \text{if } n=0 \\ A(n-1,1) & \text{if } n>0 \text{ and } m=0 \\ A(n-1,A(n,m-1)) & \text{if } n>0 \text{ and } m>0 \end{cases}$$

If we assign $\omega \times n + m$ to each recursive call, then:

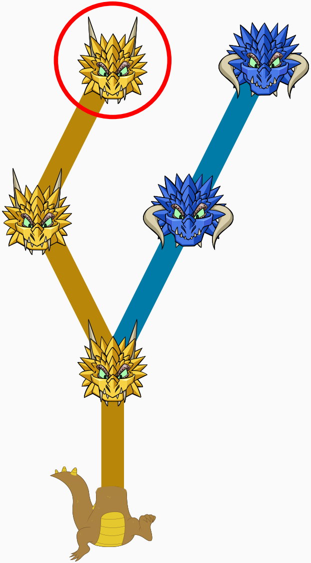
- first case: instant termination
- second case: $\omega \times n + 0 \succ \omega \times (n-1) + 1$
- third one: $\omega \times n + m - 1 < \omega \times n + m$ so the inner call terminates by induction and becomes some finite number M, and $\omega \times (n-1) + M \prec \omega \times n + m$

tools exist to assign ordinals automatically to functions

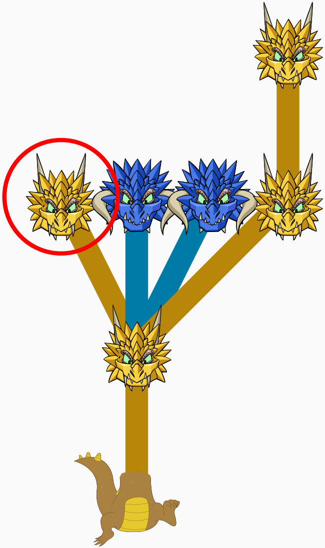
Hercules vs the Hydra

Hercules vs the Hydra

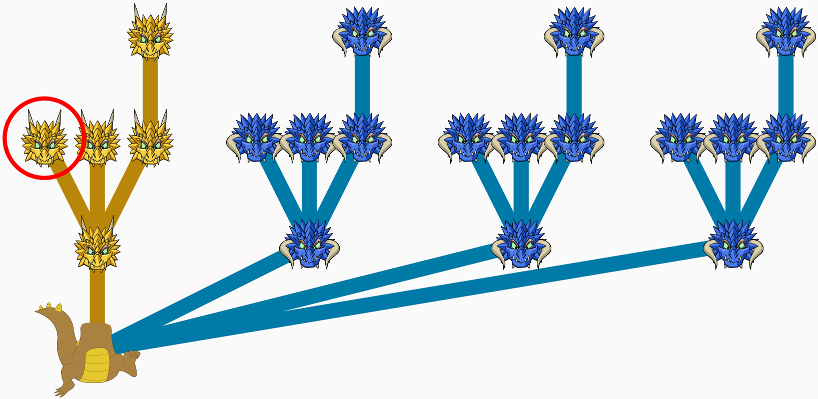
Kirby-Paris, 1982



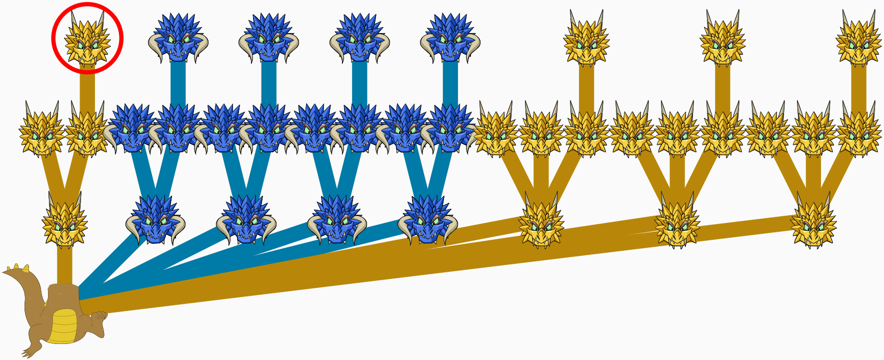
Hercules vs the Hydra



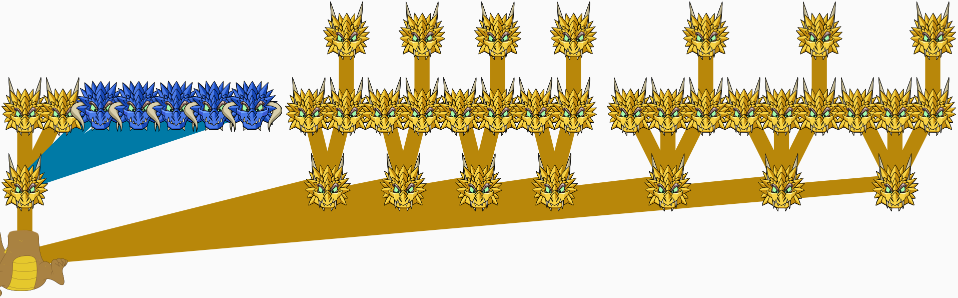
Hercules vs the Hydra



Hercules vs the Hydra



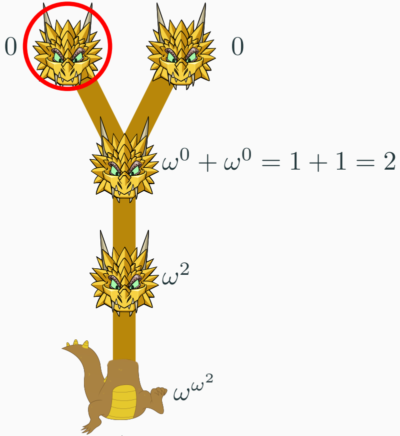
Hercules vs the Hydra



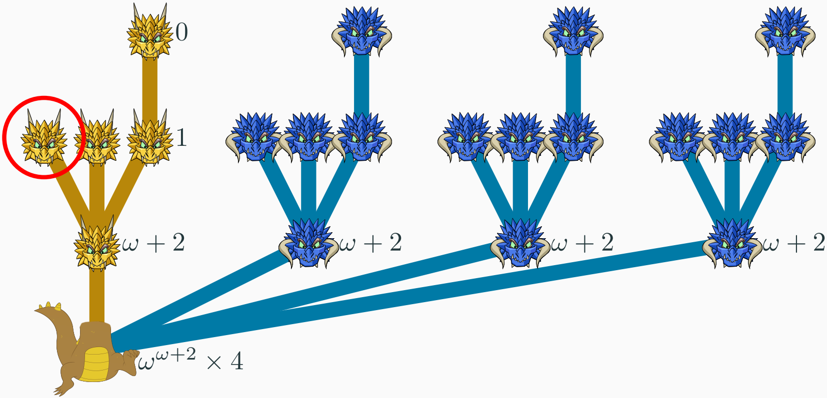
Ordinal of the Hydra

Let us assign an ordinal to each Hydra as follows:

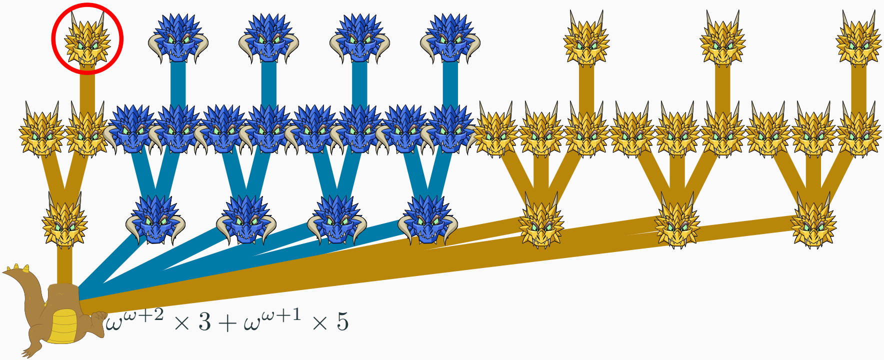
- the single-point hydra's ordinal is 0
- if the children of the Hydra have the ordinals $\alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_n$, then the Hydra gets the ordinal $\omega^{\alpha_1} + \omega^{\alpha_2} + \ldots + \omega^{\alpha_n}$



Ordinal of the Hydra



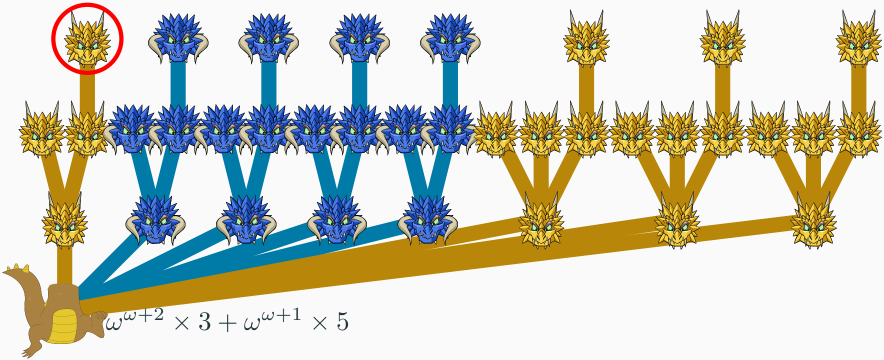
Ordinal of the Hydra



$$\omega^{\omega+2} \times 3 + \omega^{\omega+1} \times 5$$

The ordinal of the Hydra always decreases

Ordinal of the Hydra



$$\omega^{\omega+2} \times 3 + \omega^{\omega+1} \times 5$$

The ordinal of the Hydra always decreases

Hercules always wins, no matter what

How to represent order types by a finite description?

Ordinals smaller than ϵ_0 can be represented by a recursive Cantor normal form.

An idea: let us use lexicographic orderings of formal languages!

- binary words, say
- $u < v$ iff either u is a prefix of v, or $u = x0y$, $v = x1z$ for some x, y, z

- $\varepsilon < 0 < 00 < 000 < 0000 < \ldots$, so $o(0^*) = \omega$
- $\ldots < 0001 < 001 < 01 < 1$, so $o(0^*1) = -\omega$, the order type of the negative integers
- $o(0^*(0^*1 + 1^+)) = \zeta$
- $10 < 100 < 1000 < \ldots < 110 < 1100 < 11000 < \ldots < 1110 < \ldots$, so $o(1^+0^+) = \omega^2$
- $o((00 + 11)^*01) = \eta$

What order types can be represented?

Every countable order type is the (lexicographic) order type of some language over $\{0, 1\}$.

Main questions

- How can we define a language?
  - regular languages
  - context-free languages
  - context-sensitive languages — these can go well beyond ϵ_0
  - one-counter languages
- Can we work by order types given by languages at all?
  - Isomorphism problem: can we decide for two languages K and L whether $o(K) = o(L)$?
  - Can we "compute" $o(KL)$, $o(K \cup L)$ or $o(K^*)$ if we know K, L, $o(K)$ and $o(L)$ in some other representation? say, their Cantor normal form if they are ordinals

Well-ordered vs. scattered orderings

Well-orderings do not contain $-\omega$.

Scattered orderings are those not containing η.

Their order types are the scattered order types.

- Each ordinal is scattered.
- ζ is scattered.
- $\zeta \times \zeta$ is scattered.
- $\omega + (-\omega) + (-\omega) + \omega + (-\omega) + \omega + \omega + (-\omega) + \dots$ is scattered.

  there are already uncountably many from these guys
- $\{0,1\}^*$ is not scattered.

Hausdorff's theorem

Hausdorff assigned to each scattered (countable) order type a (countable) ordinal, its "rank" (intuitively, a sort of "embedding depth").

To each ordinal α, let us define a class H_α of orderings as follows:

- let H_0 contain all the finite orderings; this is an Ésik-Iván modification from 2012
- for $\alpha > 0$, let H_α be the smallest class of orderings that is
  - closed under finite sum and
  - contains all the orderings of the form $\sum_{i \in \zeta} X_i$ with each X_i being in some H_{α_i} with $\alpha_i < \alpha$.

If an ordering is a member of an H_α, let its rank be the smallest such α.

rank of an order type is defined in the expected way

Rank examples

- 0 and 1 are finite, so they have rank 0.
- $\omega = \ldots + 0 + 0 + 0 + 1 + 1 + 1 + 1 + 1 + \ldots$ is a ζ-sum of order types of rank smaller than 1, so its rank is 1.
- $\zeta = \ldots + 1 + 1 + 1 + 1 + 1 + \ldots$ is a ζ-sum of order types of rank smaller than 1, so its rank is also 1.
- $\zeta + \zeta + \omega + 1$ is a finite sum of order types of rank at most 1, so its rank is also 1.
- $\zeta \times \omega$ is an ω-sum of ζs, that is, $\ldots + 0 + 0 + 0 + \zeta + \zeta + \ldots$, a ζ-sum of summands having rank smaller than 2, so its rank is 2.
- ω^n has rank n.
- $\omega^\omega = 1 + \omega + \omega^2 + \ldots$ has rank ω.

In general, if α is an ordinal with Cantor normal form $\alpha = \omega^{\alpha_1} \times n_1 + \ldots + \omega^{\alpha_k} \times n_k$, then its rank is α_1.

Hausdorff's theorem

Exactly the (countable) scattered order types have (a countable) rank.

Hence, when reasoning over scattered order types, one technique is to use induction on its rank. well-founded induction can be used on well-ordered sets, like ordinals

Why scattered order types?

Similarly to the technique that ordinals can be used to prove termination of "one-player" systems, scattered order types can be used to prove termination of some concurrent, "two-player" systems, where the aim of one player is to terminate the system, while the other tries to make it run indefinitely.

Some results on regular/context-free order types

On regular order types

- If L_1 and L_2 are regular languages, then it is decidable whether $o(L_1) = o(L_2)$ holds.

  Bloom–Choffrut, TCS, 2001

- An ordinal is regular if and only if it is smaller than ω^ω.

  Thomas, RAIRO, 1986

- Every scattered regular order type has rank smaller than ω.

  but one cannot have all of them since there are uncountably many even for rank 2

- There is an operational characterization of scattered regular order types, involving 1, ω, $-\omega$, and the operations $+$ (binary sum), $\times\omega$ and $\times-\omega$.

  Heilbrunner, RAIRO, 1980

Some results on regular/context-free order types

On context-free order types

- It is undecidable for an input context-free language L whether $o(L) = \eta$.

  Ésik, IPL, 2011

- It is decidable whether $o(L)$ is scattered, or well-ordered.

  Bloom–Ésik, Fundamenta Informaticæ, 2010; Bloom–Ésik, FICS 2009

- The rank of each deterministic context-free scattered language is smaller than ω^ω.

  Ésik, DLT, 2011; Bloom–Ésik, IJFCS, 2011

- The deterministic context-free ordinals are exactly those smaller than ω^{ω^ω}.

  Ésik, DLT, 2011

- To each ordinal o smaller than ω^{ω^ω} there exists a so-called "ordinal grammar" G (whose nonterminals each generate a prefix-free language) with $o(G) = o$. But in general, there is no algorithm for transforming a context-free grammar generating a well-ordered language to an order equivalent ordinal grammar.

  Bloom–Ésik, Fundamenta Informaticæ, 2009

Our contributions to the field

Main contributions

- The Hausdorff-rank of context-free ordinals is less than ω^ω.

  Ésik–Iván, LATIN 2012

  Thus, exactly the ordinals smaller than ω^{ω^ω} are the context-free ones.
- If G is an ordinal grammar, then the Cantor normal form of $o(G)$ is effectively computable. Hence, the isomorphism problem of context-free ordinals is decidable if the ordinals are given by ordinal grammars.

  Gelle–Iván, TCS, 2019
- It is decidable whether $o(L)$ is a scattered context-free language of rank at most 1, and if so, then $o(L)$ is effectively computable as a finite sum of summands, each being ω, $-\omega$ and 1.

  Gelle–Iván, GandALF 2019 and Gelle–Iván, SOFSEM 2020
- The rank of a scattered one-counter language is always smaller than ω^2.

  Gelle–Iván, manuscript, submitted

Open questions

Open questions, short-term

- Characterize the scattered context-free order types of rank 2. Is their isomorphism problem decidable?
- Is there a way to compute $o(K \cup L)$ and $o(KL)$ effectively if both K and L are scattered context-free languages of known order types?

Open questions, long-term

- Is the isomorphism problem of scattered order types decidable?
- Is there an operational characterization of scattered context-free order types?

Some proof techniques

Techniques against scattered context-free languages

- If G generates a scattered language, then for each rule $A \to \alpha$ there can be at most one nonterminal B in α within the same component as A (that is, with $B \Rightarrow^* uAv$ for some u, v).
- If G generates a scattered language, then for each nonterminal X there exists a word u_X such that whenever $X \Rightarrow^* uX\beta$, then $u \in u_X^*$.
- For this definition of the rank we use, we have
  - $o(KL) = o(L) \times o(K)$ if K is prefix-free (!)
  - the rank of $o(K \cup L)$ is at most the max rank of $o(K)$ and $o(L)$
- If L^* is an infinite scattered language, then $L^* \subseteq u^*$ for some word u, hence it is a prefix chain and so $o(L^*) = \omega$

Most of our results went through by applying induction in a bottom-up way to the strongly connected components of the graph of G, and for each sentential form α at that level, reasoning about the possible order type of the language generated by α.

Thank you for your attention.