On k-term DNF with the largest number of prime implicants*

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Abstract

It is known that a k-term DNF can have at most $2^k - 1$ prime implicants and that this bound is sharp. We determine all k-term DNF having the maximal number of prime implicants. It is shown that a DNF is maximal if and only if it corresponds to a non-repeating decision tree with literals assigned to the leaves in a certain way. We also mention some related results and open problems.

1. Introduction

Prime implicants of a Boolean function, or, in other words, maximal subcubes of a subset of the n-dimensional hypercube, form a basic concept for the theory of Boolean functions and their applications. Concerning the maximal number of prime implicants, it is known that an n-variable Boolean function can have at most $O(\frac{3^n}{\sqrt{n}})$ prime implicants, and there are n-variable Boolean functions with $\Omega(\frac{3^n}{n})$ prime implicants (see, e.g., [4]).

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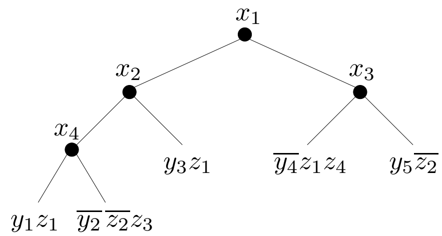

Figure 1: A non-repeating, unate-leaf decision tree (NUD)

Another case considered is the maximal number of prime implicants of Boolean functions represented by disjunctive normal forms (DNF) with a bounded number of terms. The result that a k-term DNF can have at most $2^k - 1$ prime implicants was discovered independently by Chandra and Markowsky [4], Levin [17] and McMullen and Shearer [19]. For a recent application in computational learning theory, see Hellerstein and Raghavan [9]. It was shown by Laborde [16], Levin [17] and McMullen and Shearer [19] that the bound is sharp, i.e., there are k-term DNF with $2^k - 1$ prime implicants (Chandra and Markowsky gave an example with more than $2^{k/2}$ prime implicants). In view of these results, we call a DNF *maximal* if it has k terms and $2^k - 1$ prime implicants for some k.

In this paper we complete the results of [4, 16, 17, 19] by determining all the maximal disjunctive normal forms. In order to formulate the description, let us introduce the following definition.

By a tree we mean a rooted binary tree such that for every inner node, the edge leading to its left (resp., right) child is labeled 0 (resp., 1). For a given $k \geq 2$ and $r \geq 0$, let us consider the pairwise distinct variables $x_1, \ldots, x_{k-1}, y_1, \ldots, y_k$ and $z_1, \ldots, z_r$. For each of the y and z variables, pick an orientation, i.e., form the literals $y_i^{\epsilon_i}$ ($i = 1, \ldots, k$) and $z_j^{\delta_j}$ ($j = 1, \ldots, r$), where for ϵ_i and δ_j the value 1 (resp., 0) corresponds, as usual, to an unnegated (resp., negated) variable. A *non-repeating, unate-leaf decision tree (NUD) T* over these variables and literals is constructed by taking a tree with $k - 1$ inner nodes (and thus with k leaves), assigning to each inner node a distinct x variable, assigning to each leaf a distinct y literal from those formed above, and, in addition, assigning to each leaf an arbitrary subset of the z literals formed above. The set of leaves of T is denoted by L. If we want to mention the number of x variables and y literals used in the construction, then we refer to T as a k-NUD (the value r is irrelevant). Figure 1 gives an example of a 5-NUD (the labeling of the edges is omitted for simplicity).

A k-NUD represents a k-term DNF, determined as follows. For a leaf $\ell \in L$, let the term t_ℓ be the conjunction of the x literals along the path leading to ℓ (where traversing an edge labeled 1 corresponds to an unnegated literal, and traversing an edge labeled 0 corresponds to a negated literal) and of the y and z literals assigned to ℓ. The k-term DNF represented by the k-NUD T is

$$\varphi_T = \bigvee_{\ell \in L} t_\ell.$$

For example, the 5-term DNF represented by the 5-NUD of Figure 1 is

$$\overline{x_1}\,\overline{x_2}\,\overline{x_4}\,y_1\,z_1 \;\vee\; \overline{x_1}\,\overline{x_2}\,x_4\,\overline{y_2}\,\overline{z_2}\,z_3 \;\vee\; \overline{x_1}\,x_2\,y_3\,z_1 \;\vee\; x_1\,\overline{x_3}\,\overline{y_4}\,z_1\,z_4 \;\vee x_1\,x_3\,y_5\,\overline{z_2}.$$

The Boolean function represented by φ_T can also be thought of in the following way: given a truth assignment a to all the variables, use the values of the x variables to determine a path from the root to a leaf. The function value is 1 if a makes all the y and z literals assigned to this leaf true, and it is 0 otherwise. It is clear from the definition that the input vectors accepted at a leaf ℓ are precisely those vectors which satisfy the term t_ℓ. The function φ_T is a generalized addressing function or multiplexer [20, 25]. If a DNF φ comes from a NUD T, then T can be reconstructed from φ. The y and z literals are those which are unate in φ, i.e., their negation does not occur in φ, while the x variables are those which occur both negated and unnegated. Among the x variables, the one labeling the root is the only one which occurs in every term (either unnegated or negated). The left child is the only x variable which occurs in every term containing the negation of the root variable, etc. In view of this correspondence, with some abuse of terminology, we can talk about a DNF being a NUD, rather than being equivalent to a NUD. The maximal DNF of [16, 19] (resp., [17]) corresponds to a tree which is a single path (resp., a complete binary tree), without any z literals. A NUD generalizes these examples by allowing for an binary arbitrary tree and for the additional z literals. Now we can formulate the description of maximal DNF.

Theorem 1. *A DNF is maximal if and only if it is equivalent to a NUD.*

A closely related class of DNF *tautologies* is obtained if we consider trees with the same kind of inner nodes, but without any literals assigned to the leaves. In the case of the example of Figure 1, the corresponding DNF tautology is

$$\overline{x_1}\,\overline{x_2}\,\overline{x_4} \;\vee\; \overline{x_1}\,\overline{x_2}\,x_4 \;\vee\; \overline{x_1}\,x_2 \;\vee\; x_1\,\overline{x_3} \;\vee x_1\,x_3\,.$$

Let us refer to this class of tautologies as *nonrepeating decision tree* tautologies, or *ND*'s. The main step in the proof of Theorem 1, the ND Lemma (Lemma 11) is to show that for every DNF tautology the following two properties are equivalent: *a)* any two of its terms have exactly one conflicting pair of literals (in other words, the terms are pairwise neighboring), *b)* it is an ND. Lemma 11 was proven recently, independently from our work, by Kullmann [14, 15]. Kullmann's proof uses the concept of Hermitian defect and other concepts from linear algebra. (The Hermitian rank of a symmetric matrix is the maximum of the number of positive and the number of negative eigenvalues of the matrix (Gregory, Watts and Shader [7]), and the Hermitian defect is the difference of the order of the matrix and its Hermitian rank [14, 15].) Kullmann also uses the characterization of ND's as strongly minimal tautologies with the additional property that the number of terms is one more than the number of variables (Aharoni and Linial [1], Davydov *et al.* [5], Kullmann [13]), proved using Hall's theorem or resolution techniques. (A tautology is strongly minimal if deleting any term, or adding any literal to a term results in a non-tautology.) Our proof is an elementary combinatorial argument.

We note that ND's come up in other contexts as well, e.g., in connection with the complexity of analytic tableaux (Urquhart [24], referring to earlier unpublished work of Cook, and Arai *et al.* [2]). Another related topic is the decision tree complexity of tautologies (Lovász *et al.* [18]).

The characterization of ND's as pairwise neighboring DNF tautologies is a direct consequence of the following *Splitting Lemma* (Lemma 10): if the n-dimensional hypercube is partitioned into subcubes of pairwise distance one, then there is a split of the whole cube into two half cubes such that every cube of the partition is contained in one of the two halves. We also consider the question of what can be said about cube partitions without the distance assumption. The goodness of a split into two half cubes can be measured by the fraction of the total volume of subcubes contained in one of the two halves (thus in the distance 1 case one always has a split of measure 1). This measures the fraction of points for which flipping the component corresponding to the two half cubes gives a point in a different subcube of the partition. Thus the goodness of the split measures the influence of the variable corresponding to the half cubes, on the partition (for other notions of influence, see, e.g., Hammer *et al.* [8] and Kahn *et al.* [11]). We give general lower and upper bounds for the best achievable split. The upper bound uses a result of Savický and Sgall [21] on DNF tautologies with bounded occurrences of the variables.

Recent related work on the combinatorial aspects of the satisfiability problem (see Kullmann [15] for a recent survey) makes use of the connection with partitioning complete graphs into complete bipartite graphs (bicliques). This connection, and in particular, the Graham–Pollak theorem [6] is used by Laborde [16] to show that a maximal k-term DNF contains at least $2k - 1$ variables. (This result, in turn, follows immediately from Theorem 1 above without using the Graham–Pollak theorem.) We give an application of the Splitting Lemma (Lemma 10) to show that the family of recursive partitions into complete bipartite graphs has an extremal property among all partitions into complete bipartite graphs.

The paper is organized as follows. After some preliminaries in Section 2, the results of [4, 16, 17, 19] are presented in Section 3. The proof of Theorem 1 is given in Section 4. Section 5 contains the bounds for the general splitting problem. The connection to partitions of complete graphs into complete bipartite graphs is discussed briefly in Section 6. Section 7 contains some further open problems on the number of prime implicants.

2. Preliminaries

A literal is a variable or a negated variable, a term is a conjunction (or a set) of literals, and a disjunctive normal form (DNF) is a disjunction of terms. The empty conjunction (resp. disjunction) is identically true (resp. false). It is assumed that terms do not contain both a variable and its negation. The size of a term t, denoted by $|t|$, is the number of its literals. The number of *conflicts* between two terms is the number of variables occurring unnegated in one term and negated in the other. A DNF is *disjoint* if any two of its terms have at least one conflict. We write $\psi \leq \varphi$ if every truth assignment satisfying ψ also satisfies φ, and $\psi < \varphi$ if, in addition, there is a truth assignment a with $\psi(a) = 0$ and $\varphi(a) = 1$. The set of vectors in $\{0,1\}^n$ satisfying φ are denoted by $T(\varphi)$. If t is a term then $T(t)$ is a subcube (or simply cube) in $\{0,1\}^n$, with $|T(t)| = 2^{n-|t|}$. With an abuse of notation, we usually write cube t instead of cube $T(t)$. (This is an example of switching freely between syntactic and semantic views of the same object, which occurs frequently in the paper and is, in general, useful in the study of Boolean functions.) For a literal z, the z half cube of $\{0,1\}^n$ is the $(n-1)$-dimensional subcube formed by the vectors for which z is true.

A term t is an *implicant* of a DNF $\varphi = t_1 \vee \cdots \vee t_k$ if $t \leq \varphi$. In this case we also say that φ is a *cover* of t, as the union of the cubes $T(t_i)$ covers the cube $T(t)$. Note that the variables occurring in t and φ may differ. It may be assumed *w.l.o.g.* that by a truth assignment we mean an assignment of truth values to every variable occurring in t or φ. The term t is a *prime implicant* of φ, if t is an implicant of φ, but every term obtained by deleting a literal from t is not an implicant of φ. The DNF φ is a *minimal cover* of the term t, if φ is a cover of t (i.e., t is an implicant of φ), but every DNF obtained from φ by deleting a term is not a cover of t.

Let t be a term, and $\varphi = t_1 \vee \cdots \vee t_k$ be a DNF. Every term t_i of φ can be uniquely written in the form

$$t_i = t_i' \wedge t_i'', \tag{1}$$

where t_i' contains all the literals from t_i which also occur in t, and t_i'' contains the remaining literals of t_i.

Given a DNF φ, let $Var(\varphi)$ (resp., $Lit(\varphi)$) denote the set of variables (resp., literals) occurring in any term of φ, and let

$$UL(\varphi) = \{z \in Lit(\varphi) : \bar{z} \notin Lit(\varphi)\} \tag{2}$$

be the set of *unate* literals in φ, i.e. the set of those literals occurring in φ, for which their negation does not occur in φ.

For $a \in \{0,1\}^n$, the vector $a^{(\ell)}$ is the vector obtained from a by flipping its component corresponding to the literal ℓ, e.g., for variables x_1, x_2, x_3, x_4 one has $1010^{(x_2)} = 1110$ and also $1010^{(\bar{x}_2)} = 1110$. Given $a, b \in \{0,1\}^n$, the term corresponding to the smallest subcube containing both a and b is obtained by including every literal corresponding to components where a and b agree. For example, the smallest subcube containing both 1010 and 1100 is $x_1\overline{x}_4$. The Hamming distance $d(a,b)$ of $a, b \in \{0,1\}^n$ is the number of components where a and b differ. The graph of the n-dimensional cube has $\{0,1\}^n$ as vertices, and edges (a,b) for every a, b of Hamming distance 1. The distance of two subcubes C_1 and C_2 is $\min\{d(a,b) : a \in C_1, b \in C_2\}$. Note that the distance of $T(t_1)$ and $T(t_2)$ is equal to the number of conflicts between the terms t_1 and t_2. A partition of the cube into subcubes can also be viewed as a disjoint DNF tautology. A partition of a cube into subcubes is *pairwise neighboring*, if any two subcubes in the partition have distance 1. A set of terms forms a pairwise neighboring partition if the corresponding set of cubes forms a pairwise neighboring partition.

3. Prime implicants and k-term DNF

In this section we describe the results of [4, 16, 17, 19] on prime implicants of k-term DNF. We give a complete presentation in order to make the paper self-contained, to clarify what are the consequences of the separate assumptions of being an implicant, a prime implicant, resp. a minimal cover, and to give an explicit formulation of results implicit in [16]. We use the notation introduced above in (1) and (2).

Proposition 2. *A term t is an implicant of a DNF φ if and only if $\bigvee_{i=1}^k t_i'' = 1$.*

Proof. For the $\Leftarrow$ direction, let a be a truth assignment such that $t(a) = 1$. Then $t_i'(a) = 1$ for every i and $t_i''(a) = 1$ for some i, so $t_i(a) = 1$ for some i, and thus $\varphi(a) = 1$.

For the $\Rightarrow$ direction assume $\bigvee_{i=1}^k t_i'' < 1$, i.e., $\left(\bigvee_{i=1}^k t_i''\right)(a) = 0$ for some a. The literals occurring in $\bigvee_{i=1}^k t_i''$ do not occur in t, but it may be the case that the negation of such a literal occurs in t. Let b be the truth assignment obtained from a by setting all the literals of t to 1. Then every literal in $\bigvee_{i=1}^k t_i''$ is either unchanged, or is changed to 0, thus $\left(\bigvee_{i=1}^k t_i''\right)(b) = 0$, and so $\varphi(b) = 0$. But $t(b) = 1$, contradicting the fact that t is an implicant of φ. □

Proposition 3. *If t is a prime implicant of φ then*

a) $t = \bigwedge_{i=1}^k t_i'$,

b) *every literal of t occurs in φ.*

Proof. For *a)*, it follows from the definition that $t \leq \bigwedge_{i=1}^k t_i'$. Assume that a variable x in t does not occur in any t_i. Then x does not occur in φ at all, though $\bar{x}$ may occur in some t_i''. But then t is an implicant of the disjunction of those terms in φ which do not contain $\bar{x}$, and so by deleting x from t we still get an implicant of φ. Part *b)* follows trivially from *a)*. □

Proposition 4. *If φ is a minimal cover of t then*

a) $Lit(t) \cap Lit(\varphi) = UL(\varphi)$,

b) $\bigvee_{i=1}^k t_i''$ *is a minimal cover of 1.*

Proof. For the $\subseteq$ part of *a)* note that if t contains a non-unate literal z of φ, then terms containing $\bar{z}$ can be deleted from φ and we still get a cover of t, contradicting the minimality of φ. For the $\supseteq$ part of *a)*, assume that a unate literal z is not contained in t. Then $\bar{z}\,t$ is also an implicant of φ, which is covered by the terms of φ not containing z. As these terms do not contain $\bar{z}$ either, their disjunction covers t as well, again contradicting the minimality of φ. Part *b)* follows from Proposition 2. □

Putting together Propositions 2, 3 and 4, we get the following.

Theorem 5. *If t is a prime implicant of φ and φ is a minimal cover of t, then*

a) *t is the conjunction of the literals in $UL(\varphi)$,*

b) *$\bigvee_{i=1}^{k} t_i''$ is a minimal cover of 1.* □

Theorem 6 ([4, 17, 19])**.** *Every k-term DNF has at most $2^k - 1$ prime implicants.*

Proof. Let φ be a k-term DNF and t be a prime implicant of φ. Consider a minimal set of terms of φ covering t. Then, by Theorem 5 *a)*, t is uniquely determined by this nonempty set of terms. □

The next result gives important structural information on maximal DNF's.

Theorem 7 ([16])**.** *Let $\varphi = t_1 \vee \cdots \vee t_k$ be a k-term DNF with $2^k - 1$ prime implicants, and let t be the term formed by the literals in $UL(\varphi)$.*

Then

a) *$\bigvee_{i=1}^{k} t_i''$ is a minimal cover of 1,*

b) *t_i'' and t_j'' conflict in exactly one variable, for every $1 \leq i < j \leq k$.*

Proof. By Theorems 5 and 6, every nonempty subset of the terms of φ is a minimal covering of some prime implicant of φ. Part *a)* follows by applying Theorem 5 *b)* to all the terms.

Let us consider now $\psi_{i,j} = t_i \vee t_j$. Again, this is a minimal cover of a prime implicant of φ. If t_i and t_j do not conflict in any variable, then, by Theorem 5 *a)*, the corresponding prime implicant is the term formed by all the literals in t_i and t_j. But that term is not a prime implicant. Indeed, it must be the case that $t_i \neq t_j$, and so $t_i \wedge t_j < t_i$ or $t_i \wedge t_j < t_j$. If t_i and t_j conflict in more than one variable, then we get a contradiction to Theorem 5 *b)*, as the disjunction of two terms with at least two conflicts cannot be 1. □

4. Proof of Theorem 1

In this section we prove Theorem 1: a DNF is maximal if and only if it is equivalent to a NUD.

First we consider the $\Leftarrow$ direction.

Lemma 8. *Every NUD corresponds to a maximal DNF.*

Proof. Let T be a k-NUD, and let H be a nonempty subset of its leaves. Define the term

$$t_H = UL(\{t_\ell : \ell \in H\}).$$

Let a be a truth assignment satisfying t_H. It follows by induction on the number of inner nodes evaluated, that on input a we arrive at a leaf belonging to H, and it follows from the definition of t_H that a satisfies every literal assigned to that leaf. Thus t_H is an implicant of φ_T.

Assume that we delete an x literal, say x_i^ϵ from t_H, to get the term t'. As $x_i^\epsilon \in UL(\{t_\ell : \ell \in H\})$, there is a leaf ℓ_1 belonging to H below the ϵ-child of the inner node x_i, but no leaf below the $(1-\epsilon)$-child of x_i is in H. Let a be the vector satisfying all the literals in t_{ℓ_1} and t_H, with every literal of the form $y_j^{\epsilon_j}$ not occurring in these terms set to 0. Let $b = a^{(x_i)}$. On the input b we arrive at a leaf ℓ_2 below the $(1-\epsilon)$-child of x_i. But the y literal assigned to ℓ_2 is set to 0 in b, and hence $\varphi_T(b) = 0$. On the other hand, b still satisfies t'. Thus t' is not an implicant.

Assume now that we delete a y literal, say $y_j^{\epsilon_j}$, from t_H, to get the term t'. Let ℓ be the leaf containing $y_j^{\epsilon_j}$. It follows from the definition of t_H that $\ell \in H$. Let a be a vector satisfying t_ℓ and t_H, and let $b = a^{(y_j)}$. Then the input b leads to ℓ, but as the literal $y_j^{\epsilon_j}$ has value 0 for vector b, we get $\varphi_T(b) = 0$. On the other hand, b still satisfies t'. Thus t' is not an implicant. The case when we delete a z literal, say $z_j^{\delta_j}$, from t_H is the same, except now there may be several leaves in H containing $z_j^{\delta_j}$. We can choose any such leaf, and repeat the same argument as for $y_j^{\epsilon_j}$. It again follows that the term obtained after deleting the literal is not an implicant.

Thus the term t_H is a prime implicant of φ_T. Terms corresponding to different subsets of L are different, as each leaf has its unique y literal. Hence φ_T has at least $2^k - 1$ prime implicants, and so it is maximal by Theorem 6. $\square$

The rest of this section contains the proof of the $\Rightarrow$ direction of Theorem 1.

Lemma 9. *Every maximal DNF is equivalent to a NUD.*

Proof. Let $\varphi = t_1 \vee \cdots \vee t_k$ be a k-term DNF with $2^k - 1$ prime implicants. Consider the term $t = UL(\varphi)$, and the decomposition $t_i = t_i' \wedge t_i''$ of the terms of φ with respect to t, as in (1). According to Theorem 7, the terms $t_1'', \ldots, t_k''$ form a pairwise neighboring partition over the non-unate variables occurring in φ, i.e., over $\{0,1\}^s$, where $s = |Var(\varphi) \setminus UL(\varphi)|$. The following lemma states a basic combinatorial property of pairwise neighboring partitions.

Lemma 10 (Splitting Lemma)**.** *If a set of $k \geq 2$ terms forms a pairwise neighboring partition, then there is a variable that occurs (unnegated or negated) in every term.*

Proof. We proceed by induction on the number of variables; the case of one or two variables is trivial. Let $u_1, \ldots, u_k$ be terms forming a pairwise neighboring partition of $\{0,1\}^s$.

Consider the ℓ half cube corresponding to an arbitrary literal ℓ. The restriction of $u_1, \ldots, u_k$ to the ℓ half cube is formed by deleting terms which contain the literal $\overline{\ell}$. It follows directly from the definitions that the restriction gives a pairwise neighboring partition of the ℓ half cube. If the restriction consists of a single cube then ℓ is a term of the original partition. In this case every other term of the original partition must contain $\overline{\ell}$ and we are done. Hence in what follows we may assume that the restrictions always contain at least two terms.

Applying the induction hypothesis to the pairwise neighboring partition of the $s-1$ dimensional cube obtained by deleting the component corresponding to ℓ, and deleting the literal ℓ from each of the remaining terms, it follows that there is a variable $Split(\ell)$, different from the variable of ℓ, contained (negated or unnegated) in every term covering a point in the ℓ half cube. As there are $2s$ literals and s variables, there are literals ℓ_1 and ℓ_2 such that $Split(\ell_1) = Split(\ell_2) = z$ for some variable z.

We claim that z occurs (negated or unnegated) in every term of the partition $u_1, \ldots, u_k$. If ℓ_1 is the negation of ℓ_2, then z must occur in every term and we are done; henceforth we can assume that ℓ_1 and ℓ_2 have different variables. Assume now for contradiction that z is not in every term of the partition. Let u be a term of the partition containing neither z nor $\overline{z}$, and let a be a point in u. Then a belongs to neither the ℓ_1 subcube, nor the ℓ_2 subcube.

Consider the points $a^{(\ell_1)}$ and $a^{(\ell_2)}$, covered respectively by terms u_{ℓ_1} and u_{ℓ_2} of the partition. Note that u_{ℓ_1} and u_{ℓ_2} are different. Indeed, if $u_{\ell_1} = u_{\ell_2} = u'$ then, as $a^{(\ell_1)}$ and $a^{(\ell_2)}$ differ in both their ℓ_1 and ℓ_2 components, u' contains neither ℓ_1 nor ℓ_2, and hence it covers a as well. This contradicts the definition of a.

The points $a^{(\ell_1)}$ and $a^{(\ell_2)}$ differ only in their ℓ_1 and ℓ_2 components; hence the unique conflict of the terms u_{ℓ_1} and u_{ℓ_2} is either ℓ_1 or ℓ_2. Assume without loss of generality that the conflict is ℓ_1. By definition, both u_{ℓ_1} and u_{ℓ_2} contain a z literal. As $a^{(\ell_1)}$ and $a^{(\ell_2)}$ do not conflict on z, both u_{ℓ_1} and u_{ℓ_2} contain the same z literal, say z. Thus so far we have that u_{ℓ_1} contains ℓ_1^{ϵ} and z, and u_{ℓ_2} contains $\ell_1^{1-\epsilon}$ and z, for some $\epsilon \in \{0, 1\}$.

Now consider the point $a^{(\ell_1,z)}$ covered by the term $u_{\ell_1,z}$ of the partition. As $a^{(\ell_1,z)}$ is in the ℓ_1 subcube, it contains a z literal, which must be $\overline{z}$. What is the unique conflict of u (the term covering a) and $u_{\ell_1,z}$? As $a^{(\ell_1,z)}$ and a conflict only on their ℓ_1 and z components, but u contains no z literal, it must be ℓ_1. Thus $u_{\ell_1,z}$ contains ℓ_1^{ϵ} and $\overline{z}$. But then u_{ℓ_2} and $u_{\ell_1,z}$ conflict in at least two components, a contradiction. □

The Splitting Lemma is now used to prove the characterization of nonrepeating decision tree tautologies mentioned in the introduction.

Lemma 11 (ND Lemma [14])**.** *A set of $k \geq 2$ terms forms a pairwise neighboring partition if and only if it is an ND.*

Proof. Apply Lemma 10 to the pairwise neighboring partition to get a variable x_1 occurring in every term. It must be the case that x_1 occurs both unnegated and negated, as otherwise the cubes would not cover the whole cube. If the x_1^{ϵ} half cube contains just one cube then we stop at that branch, otherwise we use the lemma again to get a variable which occurs in every subcube of the partition, belonging to the x_1^ϵ half cube, etc. In this way we get a tree, where the inner nodes are labeled with variables and there are k leaves $\ell_1, \ldots, \ell_k$ corresponding to the cubes in the partition. (The tree constructed is (the dual of) a special *search tree* in the sense of [18] for the partition.) The labels of the inner nodes are *different*, as the same label appearing twice would mean that some pair of cubes have distance at least 2. Indeed, if variable x_i occurs twice then let x_j be the variable labeling the least common ancestor of the two occurrences in the tree. By construction, there are terms containing $\bar{x}_i \, \bar{x}_j$, resp. $x_i \, x_j$. Thus the partition is an ND. □

Now we can complete the proof of Lemma 9. Lemma 11 gives a nonrepeating decision tree for the pairwise neighboring terms $t_1'', \ldots, t_k''$. We claim that by adding the literals in t_i' to the leaf ℓ_i, we get a k-NUD for φ. Consider any truth assignment a to the variables in φ. Evaluating the tree on a, we arrive at a leaf corresponding to a term t_i''. As $\varphi(a) = 1$ iff $t_i'(a) = 1$, the tree computes φ correctly. By construction, all the literals in the leaves are unate. Thus, in order to verify the NUD-ity of the tree, it only remains to show that for every leaf there is a literal which occurs only in that leaf (that literal will be its y literal). Assume that this is not the case, and every (unate) literal assigned to leaf ℓ_i occurs in some other leaf. Let x_j^ϵ be the last literal on the path leading to ℓ_i. Then $x_j^{1-\epsilon} \in UL(\varphi \setminus \{t_i\})$. We claim that $UL(\varphi \setminus \{t_i\}) \setminus \{x_j^{1-\epsilon}\}$ is an implicant of φ. Let a be a truth assignment satisfying every literal in $UL(\varphi \setminus \{t_i\}) \setminus \{x_j^{1-\epsilon}\}$, and let us evaluate the tree on a. If we arrive at a leaf other than ℓ_i, then $\varphi(a) = 1$ by construction. But $\varphi(a) = 1$ if we arrive at ℓ_i as well, as all unate literals in ℓ_i occur in other leaves, and thus they must be set to 1 in a. Thus $UL(\varphi \setminus \{t_i\})$ is not a prime implicant of φ, contradicting Theorems 5 and 6. □

5. The general splitting problem for cube partitions

According to the Splitting Lemma (Lemma 10), for every pairwise neighboring cube partition, the whole cube can be split into two halves in such a way that every cube of the partition is contained in one of the halves. In this section we consider the following question: what can be said without the pairwise neighboring property? Given an arbitrary partition of the whole cube into subcubes and a split into two halves, let us say that a cube in the partition is good, if it is contained in either one of the halves. We would like to find a split such that the good cubes contain many points.

Thus we consider the following quantities. Given a cube partition φ over the variables $x_1, \ldots, x_n$ and a variable x_j, let

$$v_{\varphi,j} = \sum \left\{ 2^{-|t|} : t \in \varphi, \;\; x_j \in t \;\, or \;\, \bar{x}_j \in t \right\}$$

be the fraction of the volume of good cubes in φ with respect to the x_j split of the cube, and let

$$\alpha_n = \min_{\varphi} \max_{1 \le j \le n} v_{\varphi,j},$$

where φ ranges over all cube partitions, or in other words, over all disjoint DNF tautologies. Note that as φ is a partition it holds that

$$\sum_{t\in\varphi} 2^{-|t|} = 1. \tag{3}$$

Theorem 12.

$$\frac{\log n - \log\log n}{n} \le \alpha_n \le O\left(n^{-\frac{1}{5}}\right).$$

Proof. Let $\varphi = t_1 \vee \cdots \vee t_r$ be a disjoint DNF tautology over the variables $x_1, \ldots, x_n$. If the term t_i contains x_j or $\bar{x}_j$, then t_i contributes $2^{-|t_i|}$ to $v_{\varphi,j}$. Thus

$$\sum_{j=1}^{n} v_{\varphi,j} = \sum_{i=1}^{r} |t_i| \cdot 2^{-|t_i|},$$

and there is a variable x_j with

$$v_{\varphi,j} \ge \frac{1}{n} \sum_{i=1}^{r} |t_i| \cdot 2^{-|t_i|}.$$

Let s denote the size of the shortest term in φ. As every term has size at least s, it follows from (3) that

$$\frac{1}{n} \sum_{i=1}^{r} |t_i| \cdot 2^{-|t_i|} \ge \frac{s}{n} \sum_{i=1}^{r} 2^{-|t_i|} = \frac{s}{n}.$$

On the other hand, for every variable x_j occurring in a shortest term t_i it holds that $v_{\varphi,j} \ge 2^{-s}$. Thus

$$\alpha_n \ge \min\left(\frac{s}{n}, 2^{-s}\right). \tag{4}$$

The lower bound then follows by taking $s = \log n - \log\log n$, for which the two terms in (4) are close to each other.

The upper bound follows from a construction of Savický and Sgall [21], providing an upper bound on the number of variable occurrences in tautological k-DNF formulas (a problem introduced by Tovey [23] and Kratochvíl, Savický and Tuza [12]). They constructed disjoint DNF tautologies over $n = 4^\ell$ variables, having 2^{3^ℓ} terms of size 3^ℓ, such that every variable occurs in at most a

$$\left(\frac{3}{4}\right)^\ell$$

fraction of the terms. The bound then follows by a direct calculation. □

We note that the upper bound of Savický and Sgall [21] has recently been improved almost optimally by Hoory and Szeider [10]. The improved constructions do not appear to improve the bound above, since the DNF constructed are not disjoint.

In view of Theorems 1 and 12 it may be of interest to consider the quantity α_n^d, which is defined as α_n, except that φ is restricted to cube partitions with pairwise distances bounded by d. In the construction of [21] the maximal distance grows linearly with n.

6. Partitions of complete graphs into complete bipartite graphs

Given a set of pairwise disjoint cubes in $\{0,1\}^n$, corresponding to terms $t_1, \ldots, t_r$, one can construct a covering

$$\mathcal{G} = \{G_1, \ldots, G_n\}$$

of the r-vertex complete graph K_r by complete bipartite graphs, where G_u has an edge connecting vertices v_i and v_j if terms t_i and t_j conflict in the variable x_u. If the set of cubes is pairwise neighboring, then this covering is a partition, as the complete bipartite graphs are edge disjoint.

Conversely, given a covering $\mathcal{G} = \{G_1, \ldots, G_n\}$ of K_r by complete bipartite graphs, we can construct a set of pairwise disjoint cubes $t_1, \ldots, t_r$ in $\{0,1\}^n$. For every G_u fix arbitrarily one of the sides as the left side. The term t_i contains x_u (resp. $\bar{x}_u$), if vertex v_i is contained in the left (resp. right) side of G_u. If $\mathcal{G}$ is a partition, then it follows that the t_i's are pairwise neighboring. The cubes thus constructed do not necessarily form a partition of $\{0,1\}^n$ (an example is given below).

The Graham–Pollak theorem [6] states that every partition of K_r into complete bipartite graphs consists of at least $r-1$ graphs. A large class of such partitions, which can be called *recursive* partitions, is obtained as follows. Take a complete bipartite graph on the whole vertex set. This 'takes care' of all edges connecting the two sides. In order to partition the remaining edges (those having both endpoints in the same side), repeat the same construction, i.e., recursively add similar partitions of the complete graphs formed by the two sides of this bipartite graph (see, e.g., [3]).

Consider a partition $\mathcal{G} = \{G_1, \ldots, G_n\}$ of K_r into complete bipartite graphs. Let the degree of a vertex v with respect to $\mathcal{G}$, denoted by $d_{\mathcal{G}}(v)$, be the number of G_i's containing v, and let the *volume* $vol(\mathcal{G})$ of the partition be defined as

$$vol(\mathcal{G}) = \sum_v 2^{-d_{\mathcal{G}}(v)}.$$

In view of the translation into a set of pairwise disjoint cubes in $\{0,1\}^n$ described above, $vol(\mathcal{G}) \leq 1$ for every $\mathcal{G}$, as $d_{\mathcal{G}}(v_i) = |t_i|$ for every $i = 1, \ldots, r$, and $vol(\mathcal{G}) = 1$ if and only if the cubes form a partition of $\{0,1\}^n$. For example,the partition of K_4 into the 3 complete bipartite graphs $(\{1\},\{3,4\})$, $(\{2\},\{1,4\})$, and $(\{3\},\{2,4\})$ (mentioned in [16]) has volume $\frac{7}{8}$. This partition of K_4 is not recursive. (It was actually this example which suggested Lemma 10.) As a corollary to the Splitting Lemma (Lemma 10) one gets the following characterization of recursive partitions. This characterization is also a direct consequence of Kullmann's [13, 14, 15] results.

Corollary 13. *A partition $\mathcal{G}$ is recursive if and only if $vol(\mathcal{G}) = 1$.*

Proof. The $\Rightarrow$ direction follows directly by induction on the number of vertices by considering the bipartite graph from $\mathcal{G}$ which contains all the vertices.

For the $\Leftarrow$ direction, one only has to note that the set of terms $t_1, \ldots, t_r$ constructed above is pairwise neighboring, and by the volume condition it is also a partition of the whole cube.

Applying Lemma 10 we get that there is a variable which occurs (unnegated or negated) in every term. This means that the corresponding bipartite graph contains all the r vertices. The remaining partitions of the two sides of this bipartite graph have total volume 2, and thus each side must have volume 1. The statement then follows by induction. □

The corollary shows that among partitions of K_r into complete bipartite graphs, recursive ones have the largest possible volume. Among the partitions of K_r into $r-1$ complete bipartite graphs, which ones have *minimal* volume?

7. Other open problems

In this paper we have discussed k-term DNF with the largest number of prime implicants. Similar results do not appear to be known for *shortest* prime implicants, i.e., prime implicants containing the smallest possible number of literals. The k-term DNF

$$x_1\bar{x}_2 \vee x_2\bar{x}_3 \vee \cdots x_{k-1}\bar{x}_k \vee x_k\bar{x}_1,$$

which is false for 0^k and 1^k, and true everywhere else, has $k(k-1)$ prime implicants, namely $x_i\bar{x}_j$ for every $i \neq j$. These prime implicants are all shortest prime implicants, as the DNF has no prime implicants consisting of a single literal. How many shortest prime implicants can a k-term DNF have in general?

Another question concerns the maximal number of prime implicants of a Boolean function which is true at a given number of points. As noted by Levin [17], every implicant is determined by the top and bottom of the corresponding subcube, in the componentwise partial ordering of the hypercube (the top and bottom may also be identical). Thus if a function is true at m points, then it has $O(m^2)$ prime implicants. It is also noted in [17] that the n-variable function which is true for vectors of weight between $\frac{n}{3}$ and $\frac{2n}{3}$, has $m^{\log 3 - o(1)}$ prime implicants. (This is the function with the largest known number of prime implicants among n-variable functions.) Thus the maximal number of prime implicants is bounded by two polynomial functions of m, and the question is to get sharper bounds.

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